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EQUIVARIANT DE RHAM THEORY AND GRAPHS*

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Abstract. Goresky, Kottwitz and MacPherson have recently shown that the computation of the equivariant cohomology ring of a G-manifold can be reduced to a computation in graph theory. This opens up the possibility that many of the fundamental theorems in equivariant de Rham theory may, on closer inspection, turn out simply to be theorems about graphs. In this paper we show that for some familiar theorems, this is indeed the case.

Introduction. This article will consist of two essentially disjoint parts. Part 1 is an exposition of (mostly) well-known results about G-manifolds. In section 1.1–1.3 we review the definition of the equivariant de Rham cohomology ring of a G-manifold and recall the statements of the two fundamental "localization theorems" in equivariant de Rham theory: the Atiyah-Bott-Berline-Vergne theorem and the Jeffrey-Kirwan theorem. In section 1.4 we discuss the "Smith" problem for G-manifolds (which is concerned with the question: Given a G-manifold with isolated fixed points, what kinds of representations can occur as isotropy representations at the fixed points?) Then in sections 1.5–1.7 we report on some very exciting recent results of Goresky-Kottwitz-MacPherson which have to do with the tie-in between "equivariant de Rham theory" and "graphs" alluded to in our title. These results show that for a large class of G-manifolds, M, with M^G finite, the equivariant cohomology ring of M is isomorphic to the equivariant cohomology ring of a pair (Γ, α), where Γ is the intersection graph of a necklace of embedded S^2's, each of which is equipped with a circle action (i.e., an axis of symmetry), and α is an "axial" function which describes the directions in which the axes of these S^2's are tilted. Finally, in section 1.8 we discuss a Morse theoretic recipe for computing the Betti numbers of M in terms of the pair (Γ, α).

The second part of this article is concerned with the combinatorial invariants of a pair (Γ, α), Γ being *any* finite simple d-valent graph and α an abstract analogue of the axial function alluded to above. In particular, for such a pair we will prove combinatorial versions of the theorems described in sections 1.2-1.3 and 1.8. These combinatorial "localization" theorems help to shed some light on the role of the localization theorems in Smith theory : From the localization theorems one can generate a lot of complicated identities among the weights of the isotropy representations. However, the question of whether one can extract from these identities any new information about the isotropy representations themselves has been an open question for a long time. Our graph theoretical results seem to indicate that one can't.

This article is the first of a series of two articles on graphs and equivariant cohomology. In the second article in this series we will discuss K-theoretical analogues of the results above and give a purely combinatorial proof of the so-called "quantization commutes with reduction" conjecture.

1.1. Equivariant de Rham theory. Let G be an n-dimensional Lie group which is compact, connected and abelian, i.e., an n-dimensional torus. Let $\mathfrak{g}$ be its

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