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THE MODULI SPACE OF COMPLEX LAGRANGIAN SUBMANIFOLDS*

N. J. HITCHIN†

1. Introduction. Developments in string theory over the past few years (e.g. [13], [5]) have focussed attention on a differential geometric structure induced on the base space of an algebraically completely integrable Hamiltonian system. This has been recently formalised by D.Freed [6] as a *special Kähler structure*.

The purpose of this paper is to provide an alternative approach to the geometry of special Kähler manifolds, one that is motivated by the desire to understand a more general situation than that afforded by integrable systems. We seek the natural geometrical structure on the moduli space M of deformations of a compact, complex Lagrangian submanifold Y in a complex Kählerian symplectic manifold X. In many respects what we do parallels the approach of an earlier paper [8] which began an investigation into the geometry of the moduli space of compact special Lagrangian submanifolds in a Calabi-Yau manifold. This was motivated by a desire to understand the geometry underpinning the Ströminger-Yau-Zaslow [14] approach to mirror symmetry. We are essentially attacking here the special case where the Calabi-Yau is hyperkähler, though we shall not need the full force of the existence of a hyperkähler metric on X.

Our viewpoint, as in [8], is to pay less attention to the holomorphic structure of the situation, and more to the symplectic one. Thus a complex Lagrangian submanifold of X can be characterized as a real submanifold on which the real and imaginary parts of the holomorphic symplectic 2-form vanish. Correspondingly, the differential geometric structure on the moduli space M is induced from a local embedding of M into $H^1(Y, \mathbf{R}) \times H^1(Y, \mathbf{R})$ which is Lagrangian with respect to two natural constant symplectic forms. We show that this "bilagrangian" condition for a submanifold of a product $V \times V$ of real symplectic vector spaces is equivalent to the structure of a special (pseudo-) Kähler metric on M. Moreover, it is easy to see from this point of view that a choice of symplectic basis of V yields the known fact that any special Kähler metric is generated by a single holomorphic function – the holomorphic prepotential.

Finally, we derive from our formalism the hyperkähler metric introduced in a string-theoretic context several years ago by Cecotti, Ferrara and Girardello [2]. It can be seen as a special case of the Legendre transform construction of Lindström and Roček [9], and yields, in the context of our moduli space, a hyperkähler metric on an open set of Markman's moduli space of Lagrangian sheaves [4].

Our approach offers a different perspective to special Kähler geometry, and in particular draws attention to a single naturally defined function ϕ which plays an important role: it is the Hamiltonian for the fundamental vector field, a potential for the Kähler metric and, with respect to one of the complex structures, a potential also for the associated hyperkähler metric.

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*Received October 29, 1998; accepted for publication February 11, 1999.
†Mathematical Institute, 24-29 St Giles, Oxford OX1 3LB, England (njh@maths.ox.ac.uk).