ASIAN J. MATH.
Vol. 3, No. 1, pp. 93–108, March 1999

005

MODULAR FORMS AND QUANTUM INVARIANTS OF 3-MANIFOLDS*

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1. Introduction. The Witten-Reshetikhin-Turaev (WRT) invariant of a compact connected oriented 3-manifold M may be formally defined by [16]

$$Z_{k+\check{c}_{\mathfrak{g}}}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{\frac{ik}{4\pi}\int_M \langle A, dA+[A,A]/3\rangle\, d\tau}\mathcal{D}A\,,$$

as a Feynman integral over an infinite dimensional moduli space of G-connections on M, in terms of additional data, namely a Lie algebra $\mathfrak{g}$ (with dual Coxeter number $\check{c}_{\mathfrak{g}}$) and a level $k \in \mathbb{N}$. In this paper we only consider WRT invariants with $\mathfrak{g} = \mathfrak{sl}_2$ and it is convenient to put $K = k+2$. This functional integral is not well-defined in a literal sense, being an integral over an infinite dimensional space on which there is no naturally defined measure. However, there are two meaningful statements which can be made about the object represented by the path integral, both of which derive from its form.

1. **TQFT**: Since the exponent in the integrand (the Chern-Simons action) is additive under gluing of manifolds, one would expect a certain gluing property to be satisfied by the value of $Z_K(M)$ with respect to the gluing together of 3-manifolds along a common boundary. Indeed, the invariant $Z_K(M)$ fits into a more general picture of invariants of arbitrary links in 3-manifolds (or more generally of slices of these, tangles in 3-manifolds with boundary) and as such describes a functor from the category of Riemann surfaces with punctures, whose morphisms are cobordisms, to the category of vector spaces ([3], [9]). When this property is formalised, one obtains the mathematical structure of a topological quantum field theory [2].
2. **PE**: The fact that the dependence on k enters only via a scaling of the exponent in the integrand leads one to expect (as there would be for complex path integrals of a similar form) the existence of a stationary phase expansion for K large of the form $Z_K(M) \sim \sum_A Z_K^A(M)$, where the terms $Z_K^A(M)$ are labelled by the stationary points of the Chern-Simons action (i.e., by the equivalence classes A of flat connections on M) and each term is the product of an asymptotic series in K^{-1}, an exponential in K (determined by the Chern-Simons number for the connection), and a power of K (determined by homological data). There is always one contribution $Z_K^0(M)$, coming from the trivial connection, which has the form of a formal power series in K^{-1}.

For integral values of K, however, there is a totally rigorous and combinatorial definition [13] of the invariant using the representation theory of the quantum group $U_\xi\mathfrak{sl}_2$ at the root of unity $\xi = e^{2\pi i/K}$. According to [11], $Z_K(M) \in \mathbb{Q}[\xi]$ for all rational homology spheres M. Moreover, one can construct a kind of perturbative expansion, for ξ tending to 1 through roots of unity of increasing order, by using the congruence properties of the coefficients of the polynomials representing $Z_K(M)$ for prime K (see [12]). What is obtained is a formal power series invariant $Z_\infty(M) \in \mathbb{Q}[[h]]$ in $h = \xi - 1$.

* Received March 31, 1999; accepted for publication April 2, 1999.
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