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GEOMETRY IN OXFORD C.1980-85

SIMON DONALDSON†

The first part of this essay comprises some brief reminiscences from my time as a research student of Sir Michael Atiyah: these will be commonplace to my contemporaries, but perhaps younger mathematicians may be less familiar with the research interests of this period. In the second part of the essay I will discuss some current research questions.

The early 1980's was a golden age for geometry in Oxford, or at least it seems so to me and probably to all who were lucky enough to be a part of the group lead by Atiyah at that time. This was a sizeable group—among the Faculty were Graeme Segal, Nigel Hitchin, Brian Steer, Glenys Luke, George Wilson and (somewhat later) Simon Salamon and Dan Quillen. Research students included Frances Kirwan, Michael Murray, Michael Pennington, Jacques Hurtubise, John Roe (all approximate contemporaries of the writer) and, a little later, Yat-Sun Poon, Henrik Pedersen, Peter Kronheimer and Peter Braam—with interweaving research interests. There were also many interactions with the equally large and active group of Mathematical Physicists in Oxford working with Roger Penrose. For us research students the weeks (at least during the short Oxford terms) revolved around Atiyah's "Geometry and Analysis" seminar, which met each Monday at 3pm. This gave a chance to hear many leading mathematicians, no doubt attracted to pass through Oxford by the presence of Atiyah, and the audience was always large. These seminars really shaped our outlook on the mathematical world. We enjoyed lectures from Bott about Witten's renowned "Quantum Mechanics" proof of the Morse inequalities, which was one of the sources for Floer's conception of Floer homology a few years later. Hirzebruch spoke about his application of the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ to line-arrangements in the plane. Among the other important developments at that time, we had several visits from Connes, who was beginning his theory of noncommutative geometry. After the seminars there would be lively discussions over tea. The most memorable of these seminars were those given by Atiyah himself, which were invariably virtuoso performances—giving, at least to this writer, a standard to aspire to ever since.

The mathematical ambience for our research interests at that time was to a large extent set up by four seminal papers of Atiyah–all related to Yang-Mills theory. In one direction was his solution (with Drinfeld, Hitchin and Manin) of the problem of finding all Yang-Mills instantons on $\mathbf{R}^4$—the ADHM construction [3]. In the early 1980's the analogous theory of "monopoles" on $\mathbf{R}^3$ was a lively topic, with beautiful constructions of Hitchin and Nahm. The spectral curves which featured in these constructions illustrated links with integrable systems, an area in which Segal and Wilson were working at that time. More generally, the whole area of "twistor geometry"; forging a link between 3-dimensional complex geometry and 4-dimensional Riemannian (or Lorentzian) geometry was very much to the fore, and of course a focus of interaction with the Penrose group. In the Riemannian case the fundamental reference is the paper of Atiyah, Hitchin and Singer [4] which *inter alia* set up the foundations for the

† Department of Mathematics, Stanford University, Stanford, CA 94305, U.S.A. (simon@math.stanford.edu).

study of the moduli spaces of Yang-Mills instantons. The 4-manifolds which admit twistor spaces are those with "self-dual" structures (the anti-self dual part of the Weyl tensor vanishes), these include the hyperkahler 4-manifolds and in particular the Ricci-flat metrics on K3 surfaces whose existence had been proved at the end of the 1970's by Yau, as a special case of his solution of the Calabi conjecture.

Another major theme stemmed from Atiyah's paper with Bott on the Yang-Mills equations over Riemann surfaces [2]. (This extraordinarily wide-ranging and many-faceted paper could virtually be used as a text book—introducing a student to diverse areas in modern geometry—as also could Atiyah's notes on the ADHM construction [1].) One of the most fruitful themes of this paper was to open up the whole area relating "complex" and "symplectic" quotients, as for example in Frances Kirwan's thesis. More generally, geometrical aspects of group actions in symplectic geometry, for example the Duistermaat-Heckmann theorem, were a great interest of Atiyah's at that time, and the subject of several memorable seminars. (Including one entitled "A generalisation of a theorem of Archimedes"—the prototype case being the action of the circle on the 2-sphere by rotations, which brings Archimedes' formula for the area of a zone in the sphere.) In another direction the Atiyah-Bott stratification of the space of connections was closely related to Segal's stratification of the the loop space of a Lie Group. Lectures by Segal on Loop Groups were another high-point of that period.

A third theme was provided by the paper of Atiyah and Jones "Topological aspects of Yang-Mills theory"[5], discussing the topology of instanton moduli spaces, and in particular intoducing the "Atiyah-Jones conjecture", that the homotopy groups stabilise, as the Pontrayagin class of the bundle increases, to those of the ambient space of all connections modulo gauge equivalence. A parallel case was that of the rational maps from the sphere to itself, which had been studied (for different reasons) at about the same time by Segal. One link between the two discussions was provided by monopoles, since Atiyah conjectured in about 1980 that the moduli spaces of monopoles should be identified with rational functions. There was considerable interest in proving these results using variational methods, and Taubes visited Oxford several times, explaining his work in this direction.

Nearly twenty years have passed since the period I have been recalling, and of course the landscape in this part of mathematics has evolved, although many of the topics mentioned above are still active areas of research and I will not attempt to summarise subsequent developments. (I should also apologise for any omissions from this brief survey.) Looking back, I think that one of the distinctive things that we learnt from Atiyah was his broad view of mathematics. Technical specialisation, as an algebraic geometer, differential geometer, topologist or whatever, was not particularly encouraged; the great thing was to explore the interaction of these different areas. Of course the influence of Mathematical Physics, partly through the Penrose group and partly through the general pre-occupation with Yang-Mills theory, was another distinctive feature—although now it has become much more familiar now, through the immense developments in the years since. Of course other mathematicians, such as Singer, Bott, Taubes—were heavily involved in this initiative, but Atiyah did a huge amount to bring these ideas into the mathematical mainstream.

In the paragraphs above, I have merely tried to give an impression of the way mathematical research appeared to a new research student, beginning in 1980. I have written elsewhere recently about the development of my own research within this ambience [6]. Perhaps it is worth saying here, however, that my own good fortune was to be able to bring some of the tools of nonlinear analysis to these areas; this kind of analysis was something of a gap in the Oxford group before, (and to some extent, this is still an area in which differential geometry in the United Kingdom is comparatively weak). However, I was only able to make much headway in this direction because of the good contacts through Atiyah with Harvard, which meant that Taubes' papers—which I used as a private course on nonlinear analysis—became available at an early stage.

The result of Narasimhan and Seshadri, which lay at the heart of the work of Atiyah and Bott [2], is now very firmly established and many different proofs have been given. Likewise for the generalisations to higher dimensions. Nevertheless there seem to be interesting questions remaining in this area, one of which I will now discuss. These questions are interesting as models for the more difficult problem of constructing constant scalar curvature Kähler metrics, in the direction of work of Lu [6], Luo [7] and Tian – although I will not say more about that problem here. Recall that the Narasimhan-Seshadri theorem says that a holomorphic bundle over a Riemann surface Σ is stable if and only if it admits a compatible connection whose curvature is a constant multiple of the identity. Here we fix an area form μ on the surface to regard the curvature of a connection as an endomorphism of the bundle. Now consider the ADHM construction of instantons in four dimensions. This produces the instanton connections, solutions of a PDE, as pull-backs of the standard connections on the tautological bundles over Grassmannians by a particular class of (real) algebraic maps from the 4-sphere to the Grassmannians. Thus the PDE is reduced to the algebraic problem of studying these special maps. In this vein, we can seek to relate the Narasimhan-Seshadri connection to connections pulled-back under suitable maps to the Grassmannian. The difference is that I shall consider a sequence of maps indexed by a parameter $k >> 0$ and the goal is to obtain the preferred connection in the limit as $k \to \infty$. Consider the Grassmannian $Gr_r(\mathbf{C}^n)$ embedded, as an adjoint orbit, in the Lie algebra $\mathfrak{su}(n)$. If $f : \Sigma \to Gr_r(\mathbf{C}^n)$ is a holomorphic map we can define the centre of mass of f in $\mathfrak{su}(n)$, integrating the push-forward of the measure μ. Let us say that the map is "balanced" if this centre of mass is 0, and that a map f is "b-stable" if its orbit under the natural action of $SL(n, \mathbf{C})$ contains a balanced representative. On the one hand, these definitions are close to familiar ideas in the study of quotient problems in finite-dimensions. We know that if we consider in place of Σ a finite set of points in the Grassmannians, then this set is stable in the algebro-geometric sense if and only if its orbit contains a balanced representative. As one application of the general moment map theory, one can show that the balanced representative in a b-stable orbit is essentially unique. On the other hand, the pull-back of the standard connection on the tautological bundle under the balanced maps form a preferred class of $U(r)$ connections over Σ. Now fix a positive line bundle ξ over Σ, whose curvature form is a multiple of μ, and let E be a rank r holomorphic bundle over Σ. For large k the bundle $E \otimes \xi^k$ is generated by its global sections or in other words its dual is pulled back from the tautological bundle over the Grassmannian. If we start with a bundle E and fix a basis in $H^0(E \otimes \xi^k)$, we associate to E a map $f_{E,k} : \Sigma \to Gr_r(\mathbf{C}^n)$, for suitable $n = n(k)$. If this map is b-stable, we get a pulled-back connection on $E \otimes \xi^k$, and hence a connection A_k on E. We may then formulate a two-part conjecture:

CONJECTURE.

(1) *The bundle E is stable if and only if the maps $f_{E,k}$ are b-stable for large k.*

(2) *If the maps $f_{E,k}$ are b-stable the connections A_k converge to the Narasimhan-Seshadri connection as $k \to \infty$.*

What I am really thinking of here is that one should try to prove *a priori* that the sequence of connections A_k converges, and thus give yet another approach to proving the Narasimhan-Seshadri theorem. (One can formulate a similar conjecture for bundles over higher-dimensional manifolds. It seems likely that the ADHM construction would be relevant here, in describing sequences of maps associated to "bubbling" phenomena.)

The rationale for this conjecture, beyond its appeal as a natural (and comparatively naive) route between the algebraic and differential geometry, is as follows. Giving a map f to the Grassmannian is the same as giving a bundle V and a basis of sections $s_1, \dots, s_n \in H^0(V)$. Given a fibre metric h on V and this basis of sections, we can define a section $\mathcal{F}_h$ of the endomorphism bundle End V by

$$\mathcal{F}_h = \sum_\alpha s_\alpha^* \otimes s_\alpha,$$

where s_α^* is the section of V^* corresponding to V under h. The fibre-metric on V pulled back from that on the universal bundle is characterised by the fact that $\mathcal{F}_h$ is a constant multiple of the identity, and the map f is balanced if and only if the L^2-inner products $\langle s_\alpha, s_\beta \rangle$ satisfy $\langle s_\alpha, s_\beta \rangle = \lambda \delta_{\alpha\beta}$ for a scalar λ, i.e. if the s_α are *orthonormal* in L^2, up to an overall scalar. Now if we start with a metric h and take any orthonormal basis s_α we get the same bundle endomorphism $\mathcal{F}_h$, that is $\mathcal{F}_h$ is an intrinsic invariant of the Hermitian holomorphic bundle (V, h) (and the fixed measure μ on Σ). In sum, we see that the balanced maps correspond precisely to the fibre metrics h on V which satisfy the equation

$$\mathcal{F}_h = \text{constant}.$$

This should be compared with the Narasimhan-Seshadri equation

$$F_h = \text{constant},$$

where F_h is the curvature of the connection defined by the metric. The motivation for the second part of the conjecture is that in the case when $V = E \otimes \xi^k$ and k is large, one expects the *global* invariant $\mathcal{F}_h$ to be well-approximated by the *local* invariant F_h. For example, if we fix a metric h_0 on E and let $\mathcal{F}_{(k)}$ be the section of End $(E) \equiv$ End $(E \otimes \xi^k)$ obtained from the induced metric on $E \otimes \xi^k$, one can show that

$$\mathcal{F}_{(k)} \sim k + \frac{i}{2\pi} F_{h_0} + C_\Sigma, \tag{1}$$

as $k \to \infty$, where $C_\Sigma = (1 - g(\Sigma))/\text{Area}(\Sigma)$. (In the case when E is a line bundle, or a direct sum of line bundles, this is a corollary of a theorem of Tian [8].)

Finally, it is a great pleasure to have this opportunity to record my thanks to Sir Michael for the immense help he has given me throughout the past 19 years – above all, for bringing into being such a splendid centre for research in Geometry in Oxford – and to send my best wishes for his 70th birthday, and many years to come.

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