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A HAPPY COLLABORATION

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1. To Michael Atiyah and Raoul Bott. The forward fundamental solution $E(x)$ of a hyperbolic differential operator is a distribution answering to a point impulse at time zero. A lacuna for E is defined as an open set inside the region of propagation in which there is no light or movement. The most famous lacuna of this kind, the inside of the forward light cone of light propagation in an even number of dimensions including time, was very familiar to me since my teacher Marcel Riesz had worked out a new theory of the wave equation (1949) that improves the classical treatise (1932) by Hadamard. Moreover, in a paper of my own (1947) concerned with some very special hyperbolic equations, I had discovered lacunas bounded by manifolds of codimension larger than one. Finally, during the academic year of 1946-47 when I was visiting the mathematics department of Princeton University, my friend Irving Segal once said: "You are interested in lacunas. Well, there are plenty of them in the last issue of the *Matematicheskii Sbornik."* He had seen Petrovski's paper (1945). I stayed up half the night in the library trying to read it but without understanding anything except for the fact that topology and algebraic geometry were involved.

2. Petrovski's paper. Petrovski considers operators $a(D)$, $D = \partial/i\partial x$ in n variables $x = (x_1, ..., x_n)$ that are homogeneous of degree m and hyperbolic with respect to a time variable $t = x_1$ in the sense that the equation $a(\xi) = 0$ has m real separate zeros in ξ_1 for all real $\xi_2, \dots, \xi_n$ not all zero. Such an operator has a unique fundamental solution $E(x)$ satisfying $a(D)E(x) = \delta(x)$ and vanishing for $t < 0$. The support of $E(x)$ spans a convex proper conoid C in x-space. The wave front surface W is defined as the intersection of C with the surface generated by $\operatorname{grad} a(\xi)$ when $a(\xi) = 0$. The wave front surface bounds C and splits it into open, connected and conical parts.

Improving on earlier work by Herglotz (1926,1928), Petrovski proves that $E(x)$ is real analytic when $x \in K - W$ where its derivatives of order $> m - n$ can be expressed as rational integrals over a certain cycle $C_{n-3}(x)$ of dimension $n - 3$ in the intersection of the complex hyperplane $X^* : (x.\zeta) = 0$ and the complex hypersurface $A^* : a(\zeta) = 0$. Its homology class is independent of x as long as x stays outside of the wave front surface. When n is even C_{n-3} is is simply the real part of $X^* \cap A^*$, a cycle that Petrovski denoted by $C_{\text{real}}(x)$. When n is odd, C_{n-3} is a differently defined cycle called $C_{\text{imag}}(x)$.

It follows from the above that if the cycle C_{n-3} is homologous to zero in the complex intersection $X^* \cap A^*$ then the fundamental solution $E(x)$ is a polynomial of degree $m - n$ in the corresponding part $\Omega \in K - W$. In particular, if $m < n$, Ω is a lacuna for $a(D)$ in the sense that the fundamental solution vanishes there. That C_{n-3} vanishes will be called the Petrovski condition in the sequel.

In case of the wave equation with $m = 2$ and $n > 2$, the cycle C_{real} is empty in conformity with the wave equation lacunas for even n. In the case $m \geq n$, the simplest example being $m = n = 2$ and a constant $E(x)$, Petrovski found no lacunas inside the propagation cone.

The main point of Petrovski's paper is that the Petrovski condition is necessary for lacunas which are stable under small variations of the polynomial $a(\zeta)$. His proof uses algebraic geometry available at the time, in particular Lefschetz's classical book

(1924) on the homology of algebraic surfaces. There it is shown that the homology in middle dimension is spanned by vanishing cycles and, if the dimension is even, an algebraic cycle. The simplest example is the case of hypersurfaces of dimension zero, i.e. a set Z of m points $z_1, \dots z_m$ in the complex plane. Here we may consider cycles $\alpha = \sum \varepsilon_j z_j$ which give a sign ε_j to the point z_j. The cycles of the form $\varepsilon_j z_j - \varepsilon_k z_k$ for which $\varepsilon_j - \varepsilon_k = 0$ are said to be vanishing since they vanish when the two points come together. If $f(z)$ is the polynomial $\prod(z - z_j)$ we may think of an abelian integral over the cycle α as the sum

$$\sum \frac{\varepsilon_j z_j}{f'(z_j)}.$$

If this sum vanishes when any two points z_j come together, α can contain no vanishing cycle and must be of the form $\pm \sum z_k$ which means that it is algebraic.

In his proof that the Petrovski condition is necessary, Petrovski first extended Lefschetz's work to hypersurfaces through a laborious by hand construction of the homology in middle dimension of non-singular hypersurfaces. In a second step it is proved in a way illustrated above that to any non-vanishing such cycle there is rational integral on which it does not vanish.

Petrovski wrote his paper during the war in the early forties when the Soviet government and Moscow University were moved from Moscow to Kuybyshev. The working conditions may be described in Petrovski's own words: only formally a university.

Most of my insight into Petrovski's paper I got from Jean Leray in the early sixties when we tried among other things to understand Petrovski's paper. Leray (1962) could later use Petrovski's cycles for an extension of the transform of Laplace to several variables.

Petrovski used only hyperbolic operators whose characteristic polynomials define non-singular varieties. In the late sixties, I returned to hyperbolic operators with singularities and discovered the usefulness of the local hyperbolicity cones defined below and could start working on a paper (1972) on local hyperbolicity.

3. The collaboration. In the spring of 1966 Michael Atiyah invited me to Oxford to give some lectures on hyperbolic equations. Another guest on that occasion was Raoul Bott. In this very inspiring company it occured to me that I could expect some help with the problem of lacunas which had rested with me since 1947. In an initial step I persuaded Michael to provide me, Raoul, and himself with photocopies of Petrovski's paper from the Bodleian Library. With this step a happy collaboration was initiated.

Already from the beginning it was clear that Petrovski's laborious homology constructions should be replaced by the new and powerful theory of sheaf cohomology. Both my companions were well versed in this field and, in addition, Atiyah was a leading specialist in algebraic geometry. Another basic result had also become available, Hironaka's resolution of singularities (1964). As it turned out, the result that we required was essentially contained in a paper (1966) by Grothendieck written in the form of a letter to Atiyah and extending Atiyah-Hodge (1955). It states that the cohomology of the complement of an affine hypersurface can be realized by rational differential forms with high enough poles on the hypersurface. Returning to the points $Z = (z_1 \dots, z_m)$ and polynomial $f(z) = \prod(z - z_k)$ above, the simplest example is the space of rational differentials $g(z)dz/f(z)$ which vanish at infinity. Any cycle in the complement of Z which is orthogonal to these forms is homologous to zero.

The homology of Z is now given by the residues of these differentials. Compared to Lefschetz's analysis of the homology of a hypersurface Z by vanishing cycles we have now moved to the homology and cohomology of its complement.

The result of our collaboration is a two part paper in Acta Mathematica, (1970, 1973) by the three of us. It carries a dedication in Russian: To Ivan Georgievich Petrovski with respect and admiration. I wrote the hyperbolic part and Atiyah the topological part that details and makes precise Grothendieck's paper by specifying lower bounds on the order of the poles. Raoul Bott contributed a vanishing theorem and played the important part of the genial companion and therefore it was decided against his vivid protests that Bott should be the third author. I have dedicated this paper to my two collaborators it is in memory of the good time that they gave me and our happy collaboration.

In what follows I will sketch the main topological results and then the basic applications to the lacuna problem that we reformulated as a condition for sharp wave fronts.

4. The topology. The topological part of the paper proves an algebraic counterpart of de Rham cohomology. One of the first basic results says that if Y is a subvariety of codimension one with normal crossings of a non-singular algebraic variety X, then the cohomology groups of (X, Y) are isomorphic to the de Rham group of of rational differentials γ and $d\gamma$ with only simple poles on the components of Y.

The main result applicable to the lacuna problem concerns the complex cohomology of the complement of a hypersurface $A : a(\xi) = 0$ in projective space P_{n-1}. Let $\omega(\xi) = \xi_1 d\xi_2 \ldots d\xi_n + \ldots$ be the standard $n-1$-form on $n-1$-spheres. If $a(\xi)$ has degree m, then every cohomology class in $H^{n-1}(P_{n-1} - A)$ is represented by a differential form

$$g(\xi)a(\xi)^{-q}\omega(\xi)$$

where $g(\xi)$ is a homogeneous polynomial whose degree $k = mq - n \geq 0$ makes the differential homogeneous. In addition, q has to be sufficiently large, depending only on m and n. When A is non-singular, it suffices that $q \geq n-1$.

More generally, if $B : b(\xi) = 0$ is a hypersurface with only normal crossings, then every cohomology class in $H^{n-1}(P_{n-1} - A \cup B)$ can be represented by a differential form

$$\frac{g(\xi)\omega(\xi)}{b(\xi)a(\xi)^q}$$

where g is a homogeneous polynomial of a degree that makes the form homogeneous and, in addition, q is sufficiently large .

5. Hyperbolicity and fundamental solutions. Let $P(D)$ be a polynomial in the derivatives $D_k = \partial/\partial x_k$ in the coordinates $x_1, \ldots, x_n$ of real n-dimensional space. A fundamental solution of E is a distribution $E(x)$ such that $P(D)E(x) = \delta(x)$. The operator $P(D)$ and the corresponding characteristic polynomial $P(\xi)$ are said to be be hyperbolic with respect to a real direction $N \neq 0$ or to be in hyp(N) if it has a fundamental solution with support in a closed cone which, apart from the origin, is contained in the half-space $(N, x) > 0$. An equivalent algebraic condition is that $P(\xi - itN) \neq 0$ for all real ξ and for t greater than some real number c. It can be shown that hyp$(N) =$ hyp$(-N)$. If a is the principal part of P then $a \in$ hyp(N) and $a(\xi + iN) \neq 0$ for all real $t \neq 0$. Here a has to be essentially real and we may assume that $a(N) > 0$. In the sequel we limit ourselves to complete polynomials depending on all variables, a property shared by a hyperbolic polynomial and its principal part.

The component of the complement of the real hypersurfacee $A : a(\xi) = 0$ that contains N is an open convex cone $\Gamma(a, N) = \Gamma(P, N)$ called the hyperbolicity cone of P and a. It has the property that $P \in \mathrm{hyp}(M)$ when $M \in \Gamma(P, N)$. For any real $\eta \in R^n$, $a(\xi)$ has a localization $a_\xi(\eta)$ defined as the first non-vanishing term in the Taylor expansion $a(\xi + \eta)$ for small η. These polynomials are in $\mathrm{hyp}(N)$ and have hyperbolicity cones $\Gamma_\xi(P, N) = \Gamma_\xi(a, N)$. When $a(\xi) \neq 0$ such a cone equals all of R^n, if $a(\xi) = 0$ but $\mathrm{grad}\, a(\xi) \neq 0$ it is a half-space. In any case it contains the positive multiples of ξ.

The dual $C(P, N)$ of $\Gamma(P, N)$, defined as all x for which $x.\Gamma(P, N) \geq 0$, is called the propagation cone of P and similarly for the local propagation cones $C_\xi(P, N)$. The reason for these names is that $P(D)$ has a unique fundamental solution $E(P, N, x)$ with support in $C(P, N)$. Moreover, this fundamental solution is real analytic in the interior of $C(P, N)$ outside a certain wave front surface $W(P, N)$ defined as the union of all local propagation cones $C_\xi(P, N)$ for $\xi \neq 0$. The fundamental solution which vanishes outside $C(P, N)$ is the distribution

$$E(P, N, x) = (2\pi)^{-n} \int e^{i(x, \xi - itN)} d\xi / P(\xi - itN) \tag{1}$$

where $t < -c$. When P reduces to its principal part a we get a fundamental solution $E(a, N, x)$ of $a(D)$ which is homogeneous of degree $m - n$. That both are fundamental solutions may be verified by integration with a test function. Letting t tend to infinity, the formula has the immediate consequence that $E(P, N, x)$ vanishes when $(x, N) < 0$. That $E(P, N, x$ vanishes outside the propagation cone follows from the fact that we may use Cauchy's theorem to replace N by any element of $\Gamma(P, N)$.

When P reduces to its principal part a, this process can be made more precise by replacing the vector $N \in \Gamma(a, N$ by a continuous vector field $v(\xi)$ such that, for every ξ, $v(\xi)$ belongs to the local propagation cone $\Gamma_\xi(a, N)$. It turns out that this construction is possible when the real plane $X = \eta : (x.\eta) = 0$ does not meet any local propagation cone, i.e. when $x \in C(a, N)$ is outside the wave front surface $W(a, N)$. Under these circumstances we may also choose $(x.v(\xi)) > 0$ for all $x \neq 0$ and choose $v(\xi)$ so small that $a(\xi + iv(\xi)) \neq 0$. By Cauchy's theorem we may then replace $\xi - itN)$ in (1) by $w(\xi) = \xi + iv(\xi)$. Finally we may also make $v(\xi)$ absolutely homogeneous of degree 1 for large ξ. In this way we have transformed (2) into an absolutely convergent integral which is an infinitely differentiable function of x outside the wave front surface.

In order to express the fundamental solution as a rational integral we have to perform a radial integration amd subtract the opposite fundamental solution $E(a, -N, x)$ which vanishes when x is in the forward propagation cone. The details of this cannot be given here, we can only describe the final result.

Let us first describe a basic chain $U(a, N, X)$ that consists of points $\xi - iv(\xi)$ where $v(\xi) \in \Gamma_\xi(a, N)$ is absolutely homogeneous of degree 1, $a(\xi - itv(\xi)) \neq 0$ when $0 < t \leq 1$, and $(x.v(\xi)) > 0$ for all $\xi \neq 0$. The chain $U(a, N, X)$ with the orientation $(x, \xi)\omega(\xi) > 0$ then determines a basic cycle $\alpha^* = \alpha^*(a, N, x)$ in projective $(n-1)$ space $Z^* = R^{n*}$ relative to the plane X^*. Here the star denotes image in projective space. The basic cycle belongs to the homology group $H_{n-1}(Z^* - A^*, X^*)$ and its boundary $\partial\alpha^*$ to $H_{n-2}(X^* - A^* \cap X^*)$. In the case of strong hyperbolicity $\partial\alpha^*$ is actually a tube around the Petrovski cycle, C_{real} when n is even and C_{imag} when n is odd.

The formulas for the derivatives $D^\nu E(a, x)$ of the fundamental solution $E(a, x) =$

$E(a, N, x)$ when $x \in C(a, x) - W(a, x)$[1] are as follows

$$D^\nu E(a, x) = i(2\pi)^{1-n} \int_{\alpha^*} \chi(i(x.\xi))\xi^\nu a(\xi)^{-1}\omega(\xi) \tag{2}$$

for non-negative homogeneity, $|\nu| \leq m - n$. and

$$D^\nu E(a, x) = (2\pi)^{-n} \int_{t_x \partial\alpha^*} \chi(i(x.\xi))\xi^\nu a(\xi)^{-1}\omega(\xi) \tag{3}$$

for negative homogeneity, $|\nu| > m - n$. Here $\chi_q(t) = t^q/q!$ when $q \geq 0$ and $\chi_q(t) = (d/dt)^{-q} \log t$ when $q < 0$ The operator t_x is the tube operation from X^* to $A^* - X^*$.

6. The Petrovski condition and sharp wave fronts. It is now obvious from the formula (3) that if $\partial\alpha^*$ is homologous to zero in $H_{n-2}(X^* - A^* \cap X^*)$ then all sufficiently high derivatives of $E(a, x)$ vanish in Ω, i.e. $E(a, x)$ is a polynomial in Ω which vanishes if $m < n$ since $E(a, N, x)$ is homogeneous of degree $m - n$. Moreover, since (3) applies to all powers of $a(D)$, our topological result shows that the vanishing of $\partial\alpha^*$ is necessary for this result to apply to the derivatives of sufficiently many fundamental solutions $E(a^j, x)$ of powers of $a(D)$.[2] The formula (2), which applies in particular to $E(a, x)$ itself when $m \geq n$, does not give a lacuna. In fact our paper proves that the cycle α^* is not homologous to zero. This answers a question by Petrovski.

In view of the results above, it is now time to replace the notion of a lacuna by those sharp and diffuse wave fronts or simply fronts. A smooth function $u(x)$ defined in some open set Ω is said to have a sharp front at a point $x \in \partial\Omega$ if, close to x, it is smooth (infinitely differentiable or real analytic) up to the boundary. A diffuse front is just the opposite. In the hyperbolic case, if Ω is a connected part of $C(a, N) - W(a, N)$, the homogeneity of the fundamental solution $E(a, N, x)$ shows that $\partial\alpha^*(x)$, $x \in \Omega$ vanishes if and only if the fundamental solution has a sharp front everywhere at the boundary of Ω. Because of the homogeneity, it suffices that the front is sharp at the origin. This fact extends to from homogeneous to inhomogeneous hyperbolic operators $P(D) = a(D) - b(D)$ with complete principal part $a(D)$. In fact,

$$E(P, N, x) = \sum_0^\infty b(D)^k E(a^{k+1}, N, x).$$

The notion of a sharp front and the Petrovski condition has an an obvious extension from the origin to any other point of the wave front surface and then the Petrovski condition assumes a local form. This is the theme of the last part of our paper.

In a footnote we surmised that the wave front surface may also be the singular support of the fundamental solution. But then we did not think of my paper (1947) which deals with the operator $P(D) = \det(\partial/\partial x_{jk})$ where the variable $x = (x_{jk})$ is an $r \times r$ hermitian matrix. This operator is hyperbolic with respect to any positive definite hermitian matrix and the corresponding propagation cone consists of non-negative matrices and the wave front surface of all hermitian matrices ≥ 0 of rank less than r. But the fundamental solution $E(P, x)$ is supported only on the matrices of rank one and hence only on a small part of the wave front surface. Simpler counterexamples appeared very soon, for instance Andersson (1970). According to Hörmander (1992) our conjecture is true when the wave front surface has at most quadratic singularities.

[1] We now drop the N from the notations

[2] It is not known if the necessity prevails without recourse to these powers.

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