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NASH RESIDUES OF SINGULAR HOLOMORPHIC FOLIATIONS*

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Interesting invariants such as the local Euler obstruction of a singular variety arise considering the Nash modification of the variety ([Br2], [M]). Also, taking a Nash type modification with respect to a coherent sheaf, we have further insight to the characteristic classes of coherent sheaves ([K], [Sc]). The purpose of this paper is to study a similar modification associated to a singular holomorphic foliation and to analyze related local invariants, namely, the Baum-Bott residues [BB] and the Nash residues, which will be introduced in this paper. As an application, we give a partial answer to the rationality conjecture of Baum and Bott (see 2.5 below).

Let $\mathcal{F}$ be a singular holomorphic foliation on a complex manifold M. The Baum-Bott residues are basically the characteristic classes of the normal sheaf $\mathcal{N}_{\mathcal{F}}$ of the foliation localized at the singular set $S(\mathcal{F})$ of $\mathcal{F}$. In section 2, we recall the Baum-Bott residues in the framework of Čech-de Rham cohomology, which is slightly different from the original one in [BB]. The method is suitable also for dealing with problems on singular varieties and is reviewed in section 1 (see also [Su2]). In a more precise way, for each compact connected component S of $S(\mathcal{F})$ and for a homogeneous symmetric polynomial of certain degree, we have the residue in the homology of S, with $\mathbb{C}$-coefficients in general. Despite their importance, not much is known about these residues. In order to analyze them, we consider, in section 3, the Nash type modification M^ν of M with respect to the foliation $\mathcal{F}$. The advantage of doing this is that the coherent sheaf $\mathcal{N}_{\mathcal{F}}$ is modified to a locally free sheaf $\mathcal{N}_{\mathcal{F}}^\nu$ on M^ν, although M^ν acquires singularity in general. For each compact connected component of the transform of $S(\mathcal{F})$, we define the residues as localized characteristic classes of $\mathcal{N}_{\mathcal{F}}^\nu$ and call them Nash residues. In the case M^ν is compact, we have the residue formula in the homology of M^ν (Theorem 3.4). This generalizes the work in [Se], where M^ν is assumed to be non-singular. We compare, in section 4, the Baum-Bott and Nash residues and prove that, for a polynomial with rational coefficients, the difference of the corresponding residues is a rational class (Theorem 4.1). This immediately implies the aforementioned answer to the rationality conjecture (Corollary 4.7). In some cases we can compute the differences of the two residues explicitly and can show how it is related to other familiar invariants.

1. Characteristic classes on singular varieties. As to the theory of characteristic classes, we use the Chern-Weil theory modified to fit in the framework of Čech-de Rham cohomology. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [Bo] and [MS]. For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration and characteristic classes in this cohomology theory are first studied in [L1-4]. See also [Su2-3] for these material. We use the notation and facts there.

(A) Poincaré and Alexander homomorphisms. Let V be an analytic subvariety of pure dimension n in a complex manifold W of dimension $m = n + k$. We set

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