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SINGULARITIES AND CHERN-WEIL THEORY, I THE LOCAL MACPHERSON FORMULA*

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Abstract. Let $\alpha : E \to F$ be a smooth bundle map between vector bundles with connection on a manifold X, and let $\Phi(\Omega)$ be a Chern-Weil characteristic form of either E or F. A notion of "geometric atomicity" for α is introduced. For any such map α we establish a canonical cohomology

$$(*) \qquad \Phi(\Omega) - \sum_{k\geq 0} \mathrm{Res}_{\Phi,k}\,[\Sigma_k(\alpha)] \;=\; dT$$

where $\Sigma_k(\alpha) = \{x \in X \,:\, \dim\ker(\alpha) = k\}$, $\mathrm{Res}_{\Phi,k}$ is a smooth residue form along $\Sigma_k(\alpha)$, and T is a canonical L^1_{loc}-form on X. When rank E = rank F, (*) can be written

$$\Phi(\Omega^F) - \Phi(\Omega^E) \;=\; \sum_{k>0} \mathrm{Res}_{\Phi,k}\,[\Sigma_k(\alpha)] + dT.$$

Normal sections of $\mathrm{Hom}(E,F)$ (those by definition which are transversal to the universal singularity sets Σ_k) are always geometrically atomic, and for such maps equation (*) expresses a classical formula of R. MacPherson at the level of forms and currents. Every real analytic map α is geometrically atomic, no matter how misbehaved its singularities. For those where each $\Sigma_k(\alpha)$ has the expected dimension, analogous formulas are established. In all cases, each term in the sum in equation (*) is a d-closed current. Proofs entail a direct application of the methods of singular connections and of finite volume flows developed by the authors.

Geometrically atomic maps prove to be generic or "typical" in all structured situations such as: direct sum mappings, tensor product mappings, mappings given by Clifford multiplication, etc. In each case the methods yield new formulas. This will be done in Part II.

0. Introduction. Some of the most useful theorems in topology are those which relate singularities of maps to topological invariants, such as Hopf's Theorem on vector fields or the Lefschetz Fixed-point Theorem. One of the most general results of this type is the beautiful formula of R. MacPherson [Mac$_*$] which relates the topology of the primary singularities of a normal smooth bundle map $\alpha : E \to F$ to characteristic classes of E and F.

In geometry the classical theory of Gauss-Chern-Weil relates topological invariants to local curvature data. Given two connections on a smooth bundle and a characteristic polynomial Φ, the theory produces a formula: $\Phi(\Omega_1) - \Phi(\Omega_2) = dT$, where Ω_i is the curvature of the i^{th} connection and T is a canonically defined smooth form. The gauge-invariant forms T are important in the study of the space of connections and they lead to well-known secondary invariants [CS], [ChS].

The aim here is to combine these results and derive MacPherson-type formulas locally on the manifold. Assume bundles E and F are equipped with metrics and connections, and let $\alpha : E \to F$ be a smooth bundle map. We shall derive formulas which explicitly express each Chern-Weil form $\Phi(\Omega)$ of E or F as a sum

$$(0.1) \qquad \Phi(\Omega) \;=\; \sum_k \mathrm{Res}_{\Phi,k}[\Sigma_k(\alpha)] + dT$$

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