ASIAN J. MATH.
Vol. 4, No. 1, pp. 115–122, March 2000

014

CHARACTERIZATIONS OF $\mathbb{P}^N$ IN ARBITRARY CHARACTERISTIC*

YASUYUKI KACHI† AND JÁNOS KOLLÁR‡

There are several results which characterize $\mathbb{P}^n$ among projective varieties. The first substantial theorem in this direction is due to [H-K57], using the topological properties of $\mathbb{CP}^n$. Later characterizations approached $\mathbb{P}^n$ as the projective variety with the most negative canonical bundle. The first such theorem is due to [K-O72] and later it was generalized in the papers [Ionescu86, Fujita87]. A characterization using the tangent bundle was given in [Mori79, Siu-Yau80]. Other characterizations using vector bundles are given in the papers [Ye-Zhang90, Sato99, Kachi99].

The results of [K-O72, Ionescu86, Fujita87] ultimately relied on the Kodaira vanishing theorem, thus they were restricted to characteristic zero. The aim of this note is to provide an argument which does not use higher cohomologies of line bundles. This makes the proofs work in arbitrary characteristic. Thus (1) generalizes [Ionescu86, Fujita87] and (2) is a characteristic free version of [K-O72]. The proofs of [Ionescu86, Fujita87] work if we know enough about extremal contractions of smooth varieties. Thus the results have been known in any characteristic for surfaces and they can be read off from the classification of contractions of smooth 3–folds in positive characteristic developed in the papers [Kollár91, S-B97, Megyesi98].

For the rest of the paper all varieties are over an algebraically closed field of arbitrary characteristic.

THEOREM 1. *Let X be a smooth projective variety of dimension n and H an ample divisor on X. Then the pair (X, H) satisfies one of the following:*
1. *$(n-1)H + K_X$ is nef.*
2. *$X \cong \mathbb{P}^n$ and H is a hyperplane section.*
3. *X is a quadric in $\mathbb{P}^{n+1}$ and H is a hyperplane section.*
4. *X is the projectivization of a rank n vector bundle over a smooth curve A and $H = \mathcal{O}(1)$.*

With a little bit of work on the scrolls in (1.4) this implies the following characterization of projective spaces and hyperquadrics.

COROLLARY 2. *Let X be a smooth projective variety of dimension n.*
1. *X is isomorphic to $\mathbb{P}^n$ iff there is an ample divisor H such that $-K_X$ is numerically equivalent to $(n+1)H$.*
2. *X is isomorphic to a hyperquadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ iff there is an ample divisor H such that $-K_X$ is numerically equivalent to nH.* □

Our proof of (1) also yields the following:

COROLLARY 3. *Let X be a smooth projective variety of dimension n. Assume that K_X has negative intersection number with some curve. Then X is isomorphic to $\mathbb{P}^n$ iff*
1. *$(-K_X \cdot C) \geq n+1$ for every rational curve $C \subset X$, and*
2. *$(-K_X)^n \geq (n+1)^n$.*

*Received March 29, 2000; accepted for publication June 2, 2000.
†Department of Mathematics, Johns Hopkins University, Baltimore MD 21218, USA (kachi@math.jhu.edu).
‡Department of Mathematics, Princeton University, Princeton NJ 08544-1000, USA (kollar@math.princeton.edu).