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THE SYMPLECTIC GEOMETRY OF POLYGONS IN HYPERBOLIC 3-SPACE*

MICHAEL KAPOVICH†, JOHN J. MILLSON‡, AND THOMAS TRELOAR§

Abstract. We study the symplectic geometry of the moduli spaces $M_r = M_r(\mathbb{H}^3)$ of closed n-gons with fixed side-lengths in hyperbolic three-space. We prove that these moduli spaces have almost canonical symplectic structures. They are the symplectic quotients of B^n by the dressing action of $SU(2)$ (here B is the subgroup of the Borel subgroup of $SL_2(\mathbb{C})$ defined below). We show that the hyperbolic Gauss map sets up a real analytic isomorphism between the spaces M_r and the weighted quotients of $(S^2)^n$ by $PSL_2(\mathbb{C})$ studied by Deligne and Mostow. We construct an integrable Hamiltonian system on M_r by bending polygons along nonintersecting diagonals. We describe angle variables and the momentum polyhedron for this system. The results of this paper are the analogues for hyperbolic space of the results of [KM2] for $M_r(\mathbb{E}^3)$, the space of n-gons with fixed side-lengths in $\mathbb{E}^3$. We prove $M_r(\mathbb{H}^3)$ and $M_r(\mathbb{E}^3)$ are symplectomorphic.

1. Introduction. An (open) n-gon P in hyperbolic space $\mathbb{H}^3$ is an ordered (n+1)-tuple $(x_1, ..., x_{n+1})$ of points in $\mathbb{H}^3$ called the vertices. We join the vertex x_i to the vertex x_{i+1} by the unique geodesic segment e_i, called the i-th edge. We let Pol_n denote the space of n-gons in $\mathbb{H}^3$. An n-gon is said to be closed if $x_{n+1} = x_1$. We let $CPol_n$ denote the space of closed n-gons. Two n-gons $P = [x_1, ..., x_{n+1}]$ and $P' = [x'_1, ..., x'_{n+1}]$ are said to be equivalent if there exists $g \in PSL_2(\mathbb{C})$ such that $gx_i = x'_i$, for all $1 \leq i \leq n+1$. We will either represent an n-gon P by its vertices or its edges, $P = [x_1, ..., x_{n+1}] = (e_1, ...e_n)$.

Let $r = (r_1, ..., r_n)$ be an n-tuple of positive numbers. This paper is concerned with the symplectic geometry of the space of closed n-gons in $\mathbb{H}^3$ such that the i-th edge e_i has side-length r_i, $1 \leq i \leq n$, modulo $PSL_2(\mathbb{C})$. We will assume in this paper (with the exception of §3) that r is not on a wall of D_n (see §2), hence M_r is a real-analytic manifold.

The starting point of this paper is (see §4)

THEOREM 1.1. *The moduli spaces M_r are the symplectic quotients obtained from the dressing action of $SU(2)$ on B^n.*

Here $B = AN$ is the subgroup of the Borel subgroup of $SL_2(\mathbb{C})$, $B = \{\left(\begin{smallmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{smallmatrix}\right) : \lambda \in \mathbb{R}_+,\ z \in \mathbb{C}\}$. B is given the Poisson Lie group structure corresponding to the Manin triple $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{b})$ with $\langle , \rangle$ on $\mathfrak{sl}(2,\mathbb{C})$ given by the imaginary part of the Killing form.

REMARK 1.2. *As a consequence of Theorem 1.1, the spaces M_r have an almost canonical symplectic structure (the symplectic structure depends on a choice of Iwasawa decomposition of $SL_2(\mathbb{C})$ or a ray in $\mathbb{H}^3$, but given two such choices, there exist (infinitely many) $g \in SL_2(\mathbb{C})$ inducing an isomorphism of the two Poisson structures).*

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†Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA (kapovich @math.utah.edu). Research partially supported by NSF grant DMS-96-26633.
‡Department of Mathematics, University of Maryland, College Park, MD 20742, USA (jjm@ math.umd.edu). Research partially supported by NSF grant DMS-98-03520.
§Department of Mathematics, University of Maryland, College Park, MD 20742, USA (txt@ math.umd.edu).