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ON EFFECTIVE NON-VANISHING AND BASE-POINT-FREENESS*

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The celebrated Kodaira vanishing theorem implies that the cohomology groups $H^p(X, K_X + H)$ vanish for $p > 0$ if X is a smooth projective variety and H is an ample divisor. It is natural to ask when $H^0(X, K_X + H)$ does not vanish.

More generally, we consider the following problem in this article. Let X be a complete normal variety, B an effective $\mathbb{R}$-divisor on X, and D a Cartier divisor on X. Assume that the pair (X, B) is KLT (log terminal), D is nef, and that $H = D - (K_X + B)$ is nef and big (cf. [9] for the terminology). By a generalization of the Kodaira vanishing theorem ([9] Theorem 1.2.5), we have $H^p(X, mD) = 0$ for any positive integer m. The problem is to find a condition on the integer m for which the non-vanishing $H^0(X, mD) \neq 0$ holds or moreover that the linear system $|mD|$ is free. By the base point free theorem ([9] Theorem 3.1.1), it is known that $|mD|$ is free for sufficiently large integer m. Fujita's freeness conjecture implies that it should be free if $m \geq \dim X + 1$. Our prediction is that $H^0(X, D) \neq 0$ always holds (Conjecture 2.1).

In §1, we shall derive a logarithmic version of the semipositivity theorem of [4] which is used as a fundamental tool for the later sections. In §2, our problem is reduced to the case where D is ample. In the rest of the paper, we consider the problem in the case $\dim X \leq 4$. In particular, we obtain a positive answer to the conjecture in the case $\dim X = 2$. We also prove an existence theorem for a 4-dimensional Fano manifold.

Let us recall the terminology. A normal variety X is said to have only *canonical* (resp. *terminal*) singularities, if the following conditions are satisfied:

(1) K_X is a $\mathbb{Q}$-Cartier divisor.

(2) For any birational morphism $\mu : Y \to X$ from a normal variety, if we write $\mu^* K_X = K_Y + B^Y$ with a $\mathbb{Q}$-divisor B^Y which is supported on the exceptional locus of μ, then all the coefficients of B^Y are non-positive (resp. negative).

Let X be a normal variety and B an effective $\mathbb{R}$-divisor on X. The pair (X, B) is said to be *LC (log canonical)* (resp. *KLT (log terminal)*, *PLT (purely log terminal)*) if the following conditions are satisfied:

(1) $K_X + B$ is an $\mathbb{R}$-Cartier divisor.

(2) For any birational morphism $\mu : Y \to X$ from a normal variety, if we write $\mu^*(K_X + B) = K_Y + B^Y$, then all the coefficients of B^Y are at most 1 (resp. strictly less than 1, at most 1 and strictly less than 1 for exceptional divisors).
In the case where B is not necessarily effective, the pair (X, B) satisfying (1) and (2) is called *subLC (sub-log canonical)* or *subKLT (sub-log terminal)*.

The pair (X, B) is said to be *properly LC* if it is LC but not KLT. For a KLT pair (X, B) and an effective $\mathbb{R}$-Cartier divisor B' on X, the *LC threshold* is defined to be the supremum of real numbers t such that $(X, B + tB')$ is still LC.

If the pair (X, B) is LC, and if an irreducible component E_j of B^Y has the coefficient 1, then the discrete valuation of the function field $\mathbb{C}(X)$ corresponding to the prime divisor E_j is called an *LC place*, and the image $\mu(E_j) \subset X$ an *LC center* for the pair (X, B). If we consider all the LC centers for all the resolutions μ for the fixed

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