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PROJECTIVITY VIA THE DUAL KÄHLER CONE - HUYBRECHTS' CRITERION*

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Abstract. In this note we give an elementary proof for a remarkable criterion due to Daniel Huybrechts for a Kähler surface to be projective. Our proof is based on Kodaira's celebrated work on surfaces.

Introduction. It was Kunihiko Kodaira who first noticed how one can distinguish projective manifolds among Kähler manifolds ([Kd1], [Kd2], [CK]). His celebrated criteria can be restated in terms of cones (cf. (1.4)) as follows:

KODAIRA'S CRITERION I ([KD1, MAIN THEOREM]). *A compact Kähler manifold is projective if and only if the Kähler cone contains an integral point.* □

KODAIRA'S CRITERION II ([CK], [KD2, THEOREM (3.1)]). *A compact (Kähler) surface is projective if and only if the positive cone contains an integral point.* □

On the other hand, one of the current tendencies in higher dimensional algebraic geometry going back to Kleiman is to study projective varieties via the duality between the ample cone and the so-called Kleiman-Mori cone, the cone of effective curves [KMM].

Quite recently, Daniel Huybrechts [Hu1] took this idea into his study of hyperkähler manifolds and found as a byproduct the following remarkable criterion to distinguish projective surfaces via the dual Kähler cone:

HUYBRECHTS' CRITERION ([HU1, REMARK 3.12 (III)]). *A compact Kähler surface is projective if and only if the dual Kähler cone contains an inner integral point. (For the precise definitions, see (1.4) in Section 1.)*

However his original proof relies on powerful but highly advanced techniques in complex analysis called Demailly's singular Morse theory and he himself asked in the same paper whether or not it is possible to prove this criterion in a more elementary way.

The first purpose of this short note is to answer his question by giving a proof based on Kodaira's Criteria I and II and more or less familiar results on surfaces found now in standard books, [Bea], [BPV]. Our proof also depends on the notion of algebraic dimension due to Kodaira [Kd2, Page 125], while it is almost free from the classification of surfaces, which is, needless to say, one of the other monumental works again due to Kodaira.

Although it is clear that the dual Kähler cone contains both the positive cone and the Kähler cone (1.5), it might not be so apparent whether or not Huybrechts' criterion is really a generalisation of Kodaira's criteria I and II, or more concretely, whether there really exists a surface whose dual Kähler cone contains an inner integral point of negative self-intersection. This was asked by Catanese [Ca].

However such surfaces indeed exist; we shall argue by looking at the dual Kähler

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