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ON A CONJECTURE OF DEMAILLY AND KOLLÁR*

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1. Introduction. If f is holomorphic in a neighborhood of a compact set K in a complex manifold, define $c_K(f)$ to be the supremum of all real numbers c such that $|f|^{-2c}$ is integrable on some neighborhood of K.

In their recent paper [DK], Demailly and Kollár made the following remarkable conjectures.

CONJECTURE A: Fix the compact set K. For every non-zero holomorphic function f and for every compact set L containing K in its interior, there is a number $\alpha = \alpha(f, K, L) > 0$ such that

$$\sup_L |g - f| < \alpha \quad \Longrightarrow \quad c_K(g) \geq c_K(f)$$

CONJECTURE B: Let

$$\mathcal{C}(n) = \{c_0(f) \ : f \text{ is holomorphic in a neighborhood of the origin of } \mathbf{C}^n\}$$

Then $\mathcal{C}(n)$ satisfies the ascending chain condition: every convergent increasing sequence in $\mathcal{C}(n)$ is stationary.

A version of Conjecture B in algebraic geometry has been formulated earlier, starting with the 1992 work of Shokurov [Sh][K1-2]. This algebraic geometric version of Conjecture B has been established in dimension $n = 2$ by Shokurov in [Sh], and in dimension $n = 3$ by Alexeev in [A]. Related conjectures and results in algebraic geometry can be found in [K1-2]. The exponents $c_0(f)$ also play an important role in the study of the existence of Kähler-Einstein metrics [CY][TY1-2][Si][T1-2] [Y1-2][DK].

In [DK], Demailly and Kollár had proved the following weaker version of Conjecture A: under the same conditions, for any $\epsilon > 0$, there exists a number $\alpha(f, K, L, \epsilon)$ such that

$$(A_\epsilon) \qquad \sup_L |g - f| < \alpha(f, K, L, \epsilon) \quad \Longrightarrow \quad c_K(g) \geq c_K(f) - \epsilon$$

In dimension $n = 2$, this last statement had also been obtained in [T2] and in [PS]. The purpose of this short note is to show that the methods of [PS] can also give the following theorem:

THEOREM. *Conjectures A and B hold when $n = 2$.*

It was already observed in [DK] that Conjecture A follows from Conjecture B combined with the weaker statement (A_ϵ). We shall nevertheless give direct separate proofs for both conjectures, since in our approach, the method of proof of Conjecture A is no different from that of (A_ϵ). It will also emerge from our proof that x is a limit

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