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SOME NEW OBSERVATION ON INVARIANT THEORY OF PLANE QUARTICS*

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1. Introduction. Let $S(n,m)$ denote the graded ring of projective invariants of an n-ary form (a homogeneous polynomial in n variables) of degree m. We are interested in the case $n=3$ and $m=4$. A ternary quartic form $F(x_0,x_1,x_2)$ defines a plane curve of genus 3 if it is nonsingular, and conversely any non-hyperelliptic curve of genus 3 can be realized as such a plane quartic via the canonical embedding, which is unique up to projective transformations. Thus the structure of the ring $S(3,4)$ is closely related to the moduli of genus 3 curves. (For general background of Invariant Theory, see e.g. [4], [13].)

More than thirty years ago ([5, Appendix]), we calculated the generating function (Poincaré series) of $S(3,4)$ and made a few guess (or conjecture?) about the structure of the graded ring $S(3,4)$. More recently, Dixmier [2] has proved the existence of a system of parameters for this ring (suggested in [5]) by exhibiting a system of seven explicit projective invariants.

In this paper, we study some close relationship of the ring $S(3,4)$ of projective invariants to another invariant theory, i.e. to the invariant theory for the Weyl groups $W(E_7)$ and $W(E_6)$ (cf. [1]). We are led to such a connection from the viewpoint of Mordell-Weil lattices ([8], [9]).

2. Formulation of main results. We consider the case of characterisitic zero. Taking

$$F(x_0,x_1,x_2)=\sum_{i_0+i_1+i_2=4} a_{i_0,i_1,i_2}x_0^{i_0}x_1^{i_1}x_2^{i_2}$$

with variable coefficients $\{a_{i_0,i_1,i_2}\}$, we may regard $S(3,4)$ as a graded subring of the polynomial ring $\mathbf{C}[a_{i_0,i_1,i_2}]$ graded by the total degree, consisting of those $I=I(F)\in\mathbf{C}[a_{i_0,i_1,i_2}]$ which are invariant under $SL(3)$. Namely, for any $g\in SL(3)$, let $(x_0',x_1',x_2')=(x_0,x_1,x_2)g$ and rewrite $F(x_0',x_1',x_2')$ as a polynomial $F'(x_0,x_1,x_2)$ in x_0,x_1,x_2:

$$F'(x_0,x_1,x_2)=\sum_{i_0+i_1+i_2=4} a'_{i_0,i_1,i_2}x_0^{i_0}x_1^{i_1}x_2^{i_2}.$$

We set $F^g=F'$. Then, by definition, we have

$$I\in S(3,4)\Longleftrightarrow I(F^g)=I(F)\ (\forall g\in SL(3)).$$

For any ternary quartic form F_0, we call the map $I\to I(F_0)$ the *evaluation map* of $S(3,4)$ at F_0.

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