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REVIEW OF GEOMETRY AND ANALYSIS*

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In this article, we shall discuss what the author considers to be important in geometry and related subjects.

Since the time of the Greek mathematicians, geometry has always been in the center of science. Scientists cannot resist explaining natural phenomena in terms of the language of geometry. Indeed, it is reasonable to consider geometric objects as parts of nature. Practically all elegant theorems in geometry have found applications in classical or modern physics. In order to understand the future of geometry, it is perhaps useful to review what was known in the past. Clearly what I consider to be important may not be viewed to be so by others. Also, we should always keep in mind that what is fashionable now may not be so tomorrow.

A theory can be judged to be successful only if its consequences help us understand the basic structure or the beauty of geometry.

While we shall divide the subject into several categories, the division is artificial, as the development of each section depends on other sections heavily.

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I. Submanifolds. Many very important questions in classical geometry remain unanswered. Descriptions of surfaces in Euclidean three space have been important in the subjects of computer graphics and data compression. They also play an important role in modern filmmaking.

Indeed, the theory of surfaces in three space is a central subject in geometry. Many difficult questions remain unanswered. Starting from the time of Gauss [63], geometers have always been interested in the interplay between the intrinsic internal metric structure of surfaces and their extrinsic geometry in the ambient space.

A. Isometric embedding of surfaces. An important well-known question is to characterize those intrinsic metrics on a surface which can be realized as embeddings into three space. Minkowski made the first major progress on this problem by proving that any convex polyhedron can be so realized [152]. For smooth surfaces with positive curvature, this problem is called the Weyl problem, since Weyl found the first significant estimate for the problem [221]. H. Lewy solved the problem in the real analytic category [124], and Pogorelov [169] and Nirenberg [162] solved it in the smooth category. There is also recent work by Y.Y. Li and P. Guan [85].

An important reason that the Weyl problem can be solved is that its solutions must be unique–a theorem due to Cohn-Vossen [46] and Pogorelov [168]. Uniqueness here means that any isometric embedding of the surface is related to any other by a

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