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EXCEPTIONAL MODULI PROBLEMS II*

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1. Introduction, Notation. In this note we attempt to formulate moduli problems connected with some arithmetic quotients of each of the four irreducible (real) rank 3 hermitian symmetric tube domains corresponding to the four $\mathbf{R}$-division-algebras $\mathbf{R}, \mathbf{C}, \mathbf{H}$, and $\mathcal{C}$: the reals,complex numbers, Hamilton quaternions, and Cayley division algebra respectively. The ideas in this note were largely inspired by the work of Professors Kunihiko Kodaira and Donald Spencer on the moduli of algebraic varieties [K], [KS].

First some notation. Let $V \cong \mathbf{R}^n$ be an $n-$dimensional real vector space and $\mathcal{P}$ be a self-adjoint homogeneous convex cone such that the tube domain $\mathcal{T} = V + i\mathcal{P}$ is a complex Hermitian symmetric domain whose group G of holomorphic automorphisms is a semi-simple real algebraic Lie group defined over $\mathbf{Q}$. Let $\Gamma \hookrightarrow G(\mathbf{Q})$ be an arithmetic subgroup of $G(\mathbf{Q})$.

In the following discussion, V will most frequently be one of the four Jordan algebras $\mathcal{J}_D$ of 3-by-3 hermitian matrices over $D = \mathbf{R}, \mathbf{C}, \mathbf{H}$, or $\mathcal{C}$, each of which carries a standard, positive involution $a \mapsto \overline{a}$, and $\mathcal{P}_D$ will be the cone of strictly positive elements of $\mathcal{J}_D$. Thus,

$$\mathcal{T}_D = \mathcal{J}_D + i\mathcal{P}_D \subset \mathbf{C}^{n_D}$$

is one of the four rank 3 irreducible hermitian symmetric domains and $n_{\mathbf{R}} = 6, n_{\mathbf{C}} = 9, n_{\mathbf{H}} = 15, n_{\mathcal{C}} = 27$.

Now if $\mathcal{T} = \mathcal{H}_n$ is the Siegel upper half-space and $\Gamma = \Gamma_n$, the Siegel modular group of degree n, then the orbits of Γ_n in $\mathcal{H}_n$ correspond one-to-one to the isomorphism classes of principally polarized Abelian varieties (p.p.A.v's) of dimension n. Moreover [Ba2], there is a Γ_n-invariant complex analytic closed subset $\mathcal{J}_n$ of $\mathcal{H}_n$ and a Zariski-open, Γ_n-invariant subset $\mathcal{J}_n^{\#}$ of it such that the space of orbits of Γ_n in $\mathcal{J}_n^{\#}$ is in one-to-one correspondence with isomorphism classes of canonically polarized Jacobian varieties of curves of genus n, hence, via Torelli's theorem, with the isomorphism classes of non-singular curves of genus n. If $n = 3$, then $\mathcal{J}_3 = \mathcal{H}_{\mathbf{R}} = \mathcal{H}_3$, and this special case is important for the considerations which follow.

Now it can be proved [Ba1] that there is a certain "nice" unicuspidal arithmetic subgroup $\Gamma_{\mathcal{C}}$ of $Hol(\mathcal{T}_{\mathcal{C}})$ for which the associated Eisenstein series have rational Fourier coefficients, thus the quasi-projective variety $\mathcal{T}_{\mathcal{C}}/\Gamma_{\mathcal{C}}$ is defined over $\mathbf{Q}$. However, I do not know of any interpretation of the space of orbits of $\Gamma_{\mathcal{C}}$ in $\mathcal{T}_{\mathcal{C}}$ as the space of moduli of some family of polarized algebraic varieties (possibly with additional structures). The main problem to be discussed here is how to determine such a family if one exists.

2. Approach. There is a quite remarkable coincidence, so far unexplained in satisfactory depth, between these four symmetric tube domains and the four Severi varieties of Zak et al. [LaZ], S_n, $n = 1, 2, 3, 4$, where $\dim(S_n) = 2^n$. These are given explicitly as follows (here $\mathbf{P}^n$ is the projective space of dimension n):

$$S_1 = \mathbf{P}^2 \hookrightarrow \mathbf{P}^5 (\text{Veronese imbedding});$$

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