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BIRATIONAL INVOLUTIONS OF $\mathbf{P}^2$*

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Introduction. This paper is devoted to the classification of the elements of order 2 in the group $\mathrm{Bir}\,\mathbf{P}^2$ of birational automorphisms of $\mathbf{P}^2$, up to conjugacy. This is a classical problem, which seems to have been considered first by Bertini [Be]. Bertini's proof is generally considered as incomplete, as well as several other proofs which followed. We refer to the introduction of [C-E] for a more detailed story and for an acceptable proof. However the result itself, as stated by these authors, is not fully satisfactory: since they do not exclude singular fixed curves, their classification is somewhat redundant.

We propose in this paper a different approach, which provides a precise and complete classification. It is based on the simple observation that any birational involution of $\mathbf{P}^2$ is conjugate, via an appropriate birational isomorphism $S \overset{\sim}{\dashrightarrow} \mathbf{P}^2$, to a *biregular* involution σ of a rational surface S. We are thus reduced to the birational classification of the pairs (S,σ), a problem very similar to the birational classification of *real* surfaces. This classification has been achieved by classical geometers [C]; the case of surfaces with a finite group of automorphisms has been treated more recently along the same lines by Manin[1] [Ma].

These questions have been greatly simplified in the early 80's by the introduction of Mori theory. In our case a direct application of this theory shows that the minimal pairs (S,σ) fall into two categories, those which admit a σ-invariant base-point free pencil of rational curves, and those with $\mathrm{rk}\ \mathrm{Pic}(S)^\sigma = 1$. The first case leads to the so-called De Jonquières involutions; in the second case an easy lattice-theoretic argument shows that the only new possibilities are the celebrated Geiser and Bertini involutions. Any birational involution is therefore conjugate to one (and only one) of these three types.

1. Biregular involutions of rational surfaces.

We work over an algebraically closed field k of characteristic $\neq 2$. By a *surface* we mean a smooth, projective, connected surface over k.

We consider pairs (S,σ) where S is a rational surface and σ a non-trivial biregular involution of S. We will say that (S,σ) is *minimal* if any birational morphism $f : S \to S'$ such that there exists a biregular involution σ' of S' with $f\circ\sigma = \sigma'\circ f$ is an isomorphism.

Recall that an *exceptional curve* E on a surface S is a smooth rational curve with $E^2 = -1$.

LEMMA 1.1 *The pair (S,σ) is minimal if and only if each exceptional curve E on S satisfies $\sigma E \neq E$ and $E\cap\sigma E \neq \varnothing$.*

Proof. If S contains an exceptional curve E with $\sigma E = E$ (resp. $E\cap\sigma E = \varnothing$), consider the surface S' obtained by blowing down E (resp. $E\cup\sigma E$); then σ induces

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1 We are indebted to I. Dolgachev for this reference.