ASIAN J. MATH.
Vol. 4, No. 1, pp. 19–36, March 2000

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LAGRANGIAN SUBVARIETIES OF ABELIAN FOURFOLDS*

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1. Introduction. Let $(\mathcal{W}, \omega)$ be a smooth projective algebraic variety of dimension $2n$ over $\mathbb{C}$ together with a holomorphic $(2,0)$-form of maximal rank $2n$. A subvariety $X \subset \mathcal{W}$ is called *weakly lagrangian* if $\dim X \leq n$ and if the restriction of ω to X is trivial (notice that X can be singular). An n-dimensional subvariety $X \subset \mathcal{W}$ with this property is called *lagrangian*. For example, any curve C contained in a K3 or abelian surface S is lagrangian. Further examples of lagrangian subvarieties are obtained by taking a curve $C \subset S$ and by considering the corresponding symmetric products. Alternatively, one could look at a product of different curves (of genus > 1) inside a product of abelian varieties. We will say that a variety $X \subset \mathcal{W}$ is *fibered* if it admits a dominant map onto a curve of genus > 1. In this note we construct examples of nonfibered lagrangian surfaces in abelian varieties. Our motivation comes from the following

PROBLEM 1.1. *Find examples of projective surfaces with a nontrivial fundamental group. In particular, find examples where the fundamental group has a nontrivial nilpotent tower.*

If X is fibered over a curve C of genus > 1 then the fundamental group $\pi_1(X)$ surjects onto a subgroup of finite index in $\pi_1(C)$ and consequently both $\pi_1(X)$ and its nilpotent tower are big. Therefore, we are interested in examples of surfaces where the nontriviality of $\pi_1(X)$ is not induced from curves. Consider the map to the Albanese variety $\mathrm{alb} : X \to \mathrm{Alb}(X)$. One is interested in situations where the natural map

$$H^2(\mathrm{Alb}(X), \mathbb{C}) \to H^2(X, \mathbb{C})$$

has a nontrivial kernel – the triviality of the kernel implies the triviality of the nilpotent tower (tensor $\mathbb{Q}$). Such examples were given by Campana ([3], Cor. 1.2) and Sommese-Van de Ven ([7]). However, there the kernel was found in the map

$$\mathrm{Pic}(\mathrm{Alb}(X)) \to \mathrm{Pic}(X).$$

In their construction the fundamental group of the variety is a central extension of an abelian group (and the lower central series has only two steps). The lagrangian property of $\mathrm{alb}(X) \subset \mathrm{Alb}(X)$ implies that there is a nontrivial kernel in $H^{2,0}$ (rather than on the level of the Picard groups). We produce an infinite series of surfaces X of different topological types which are contained in abelian varieties and are lagrangian (or weakly lagrangian) with respect to a nondegenerate $(2,0)$-form. The weakly lagrangian property for a subvariety X of an abelian variety A is related to the nontriviality of the fundamental group $\pi_1(X)$ as follows: the number of linearly independent generators of the second quotient of the central series of $\pi_1(X)$ is bounded from below by the dimension of the rational envelope of the space of those $(2,0)$-forms on A which restrict trivially to X.

*Received April 4, 2000; accepted for publication June 7, 2000.
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