ASIAN J. MATH.
Vol. 4, No. 2, pp. 331–336, June 2000
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REMARKS ON THE INVERSE MEAN CURVATURE FLOW*

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Abstract. In this short note we investigate the regularity of 2-surfaces evolving by its inverse mean curvature in an asymptotically flat Riemannian 3-manifold and derive an a-priori bound for the second fundamental form in terms of a quantity depending on the mean curvature and the elapsed time. This partially solves one of the 5 questions posed in [3]. The proof relies on the special geometry of asymptotically flat Riemannian manifolds and on the fact that the dimension of the evolving surface is 2.

1. Introduction. Assume that $F_t : M^n \to N^{n+1}$ is a smooth family of immersions into a smooth Riemannian manifold N evolving by the inverse mean curvature flow (IMCF)

$$\frac{d}{dt}F = \frac{1}{H^2}\overrightarrow{H}, \tag{1}$$

where $\overrightarrow{H}$ is the outward pointing mean curvature vector and $H = |\overrightarrow{H}|$. In [3] the proof of the Penrose inequality [7] for a Riemannian 3-manifold N^3 of nonnegative scalar curvature has been announced. The proof is based on a weak formulation of the IMCF and will appear in [4]. The weak solution remains equal to the smooth solution until the latter ceases to exist or $M_t := F_t(M^n)$ ceases to be a minimizing hull, which may happen sooner. In this paper we investigate the regularity of the classical solution of (1). From (4) (see below) and the maximum principle it easily follows that H admits an upper bound that depends only on the initial immersion $F_0 : M^n \to N^{n+1}$. A more important and delicate question is whether there exists an upper bound for the full second fundamental form A. Let us compare the situation to the mean curvature flow $\frac{d}{dt}F = -\overrightarrow{H}$. In [2] it was shown that a bound on the second fundamental form A implies bounds on all covariant derivatives $|\nabla^k A|^2, \forall k \geq 0$ and consequently a smooth solution for the mean curvature flow exists as long as A stays bounded. In special situations, as for example for mean convex solutions in the euclidean space, it is then possible to control the second fundamental form in terms of H, i.e. in terms of the speed alone. However, when working in arbitrary ambient spaces, this is no longer true. The aim of this paper is to derive similar results for the inverse mean curvature flow, i.e. we would like to control the second fundamental form and its covariant derivatives in terms of quantities that depend on the speed $\frac{1}{H}$. It turns out that we can do this, if we restrict the dimension to 2. Interestingly enough, this is just the important case in general relativity and moreover we are able to bound A in terms of the speed and the elapsed time when working in arbitrary asymptotically flat Riemannian manifolds N. Throughout this paper we will (somehow more generally) assume that N is a complete Riemannian 3-manifold such that all curvature quantities and covariant derivatives of arbitrary order are bounded. Let us define two constants a and b by

$$a := \max_{M_0} |A|^2$$
$$b := \max_{M_0} H^2.$$

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