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THE SYMPLECTIC FLOER HOMOLOGY OF THE FIGURE EIGHT KNOT*

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Abstract. In this paper, we compute the symplectic Floer homology of the figure eight knot. This provides first nontrivial knot with trivial symplectic Floer homology.

1. Introduction. In [4], we generalized the Casson-Lin invariant [6] to the symplectic theory point of view. Our symplectic Floer homology of knots serves a new invariant for knots, and its Euler characteristic is half of the signature of knots.

We showed that the symplectic Floer homology of the unknotted knot is trivial in [4]. The natural question arises as whether there is a nontrivial knot with trivial symplectic Floer homology. We answer this question in this paper by computing the symplectic Floer homology of the figure eight knot.

Although we know that the signature of the figure eight knot $4_1 = \overline{\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}}$ is zero, the signature does not suffice to give the information of our finer invariant - the symplectic Floer homology. For the square knot, we computed in [5] that the symplectic Floer homology is nontrivial even though its signature is zero. Our main result is the following.

THEOREM. *The symplectic Floer homology of the figure eight knot* $4_1 = \overline{\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}}$ *is*

$$HF_i^{sym}(4_1) = CF_i^{sym}(4_1) = 0, \quad \textit{for all } i \in \mathbf{Z}_{2N}.$$

To our knowledge, this is the first trivial symplectic Floer homology for the nontrivial knot. It is still an open question if there is a non-homotopy 3-sphere with trivial instanton Floer homology. We wish to build the relation between our symplectic Floer homology of knots [4] and the instanton Floer homology of homology 3-spheres [1] through the Dehn surgery technique. Using the calculation of the figure eight knot, we hope to unfold the question about vanishing instanton Floer homology for a nontrivial homology 3-sphere.

2. The symplectic Floer homology.

2.1. The symplectic Floer homology of braids. We briefly recall our definition of the Floer homology of braids in this subsection. See [4] for more details.

For any knot $K = \overline{\beta}$ with $\beta \in B_n$, the braid group, the space $\mathcal{R}(S^2 \setminus K)^{[i]}$ can be identified with the space of $2n$ matrices $X_1 \cdots, X_n, Y_1, \cdots, Y_n$ in $SU(2)$ satisfying

$$\text{(2.1)} \qquad \operatorname{tr}(X_i) = \operatorname{tr}(Y_i) = 0, \quad \text{for } i = 1, \cdots, n,$$

$$\text{(2.2)} \qquad X_1 \cdot X_2 \cdots X_n = Y_1 \cdot Y_2 \cdots Y_n.$$

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