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DENSITY OF RATIONAL POINTS ON ELLIPTIC K3 SURFACES*

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1. Introduction. Let X be a smooth projective algebraic variety defined over a number field K. We will say that rational points on X are potentially dense if there exists a finite extension K'/K such that the set $X(K')$ of K'-rational points is Zariski dense. What are possible strategies to propagate rational points on an algebraic variety? We thought of two: using the group of automorphisms $\mathrm{Aut}(X)$ and using additional geometric structures - like elliptic fibrations. The class of K3 surfaces is an ideal test case for both methods.
One of our main results is:

THEOREM 1.1. *Let X be a K3 surface defined over a number field K. Assume that X has a structure of an elliptic fibration or an infinite group of automorphisms. Then rational points on X are potentially dense.*

Here is a more detailed list of what we learned: We don't know if rational points are potentially dense on a general K3 surface with Picard group $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}$. In particular, we don't know if rational points are dense on a double cover of $\mathbb{P}^2$ ramified in a general curve of degree 6. However, we can prove potential density for a *divisor* in the space of all such K3 surfaces, corresponding to the case when, for example, the ramification curve is singular (cf. [6]). The overall picture is similar. In any moduli family of algebraic K3 surfaces we can find some union of algebraic subsets, including a divisor, such that rational points are potentially dense on the K3 surfaces corresponding to the points of this subset. More precisely,

THEOREM 1.2. *Let X be a K3 surface, defined over a number field K. Assume that* $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 2$ *and that X does not contain a (-2)-curve. Then rational points on X are potentially dense.*

REMARK 1.3. If $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 2$ and if X does not contain a (-2)-curve then either it has an elliptic fibration or it has an infinite automorphism group (but not both!). For example, a quartic surface in $\mathbb{P}^3$ containing a smooth curve of genus 2 and degree 6 doesn't admit any elliptic fibrations, but the group $\mathrm{Aut}(X_{\mathbb{C}})$ is infinite (cf. [24] p. 583, [27]).

THEOREM 1.4. *Let X be a K3 surface over K with* $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) \geq 3$. *Then rational points on X are potentially dense, with a possible exception of 8 isomorphy classes of lattices* $\mathrm{Pic}(X_{\mathbb{C}})$.

REMARK 1.5. If $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 3$ then there are only 6 types of lattices where we can't prove potential density. There are only 2 types when $\mathrm{rkPic}(X_{\mathbb{C}}) = 4$. Potential density holds for all K3 with $\mathrm{rkPic}(X_{\mathbb{C}}) \geq 5$. All K3 surfaces with $\mathrm{rk}\,\mathrm{Pic}(X_{\mathbb{C}}) = 20$ have infinite groups of automorphisms. We use Nikulin's classification of lattices of algebraic K3 surfaces (cf. [23], [22]).

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