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PERIMETER-MINIMIZING CURVES AND SURFACES IN $\mathbf{R}^n$ ENCLOSING PRESCRIBED MULTI-VOLUME*

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Abstract. Planar curves minimizing length for given area are classically characterized as circular arcs. We give a new generalization to $\mathbf{R}^n$ of such area constraints and characterize the minimizing curves. We also consider surfaces satisfying new generalized volume constraints.

1. Introduction. A smooth oriented hypersurface S in $\mathbf{R}^n$ which minimizes area for given boundary and enclosed volume has constant mean curvature. (Given volume means that for any competitor S', $S' - S$ encloses net algebraic volume 0. Alternatively, $\int_S x_1 \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$ is prescribed.) Conversely, any small portion of a smooth, constant-mean-curvature hypersurface minimizes area for given boundary and volumes [M4, Rmk. p. 76].

This paper considers m-dimensional surfaces S in $\mathbf{R}^n$ for arbitrary $m < n$ which minimize area for given boundary and "multi-volume": volumes enclosed by orthogonal projections onto all $(m+1)$-planes or just onto axis $(m+1)$-planes, which can be expressed as a multivector in the exterior algebra $\Lambda_{m+1}\mathbf{R}^n$. Of course the volumes of projections onto axis planes may be arbitrarily adjusted by adding spheres in those planes. Theorems 2.1 and 2.3 show that a minimizer exists among the generalized surfaces (rectifiable currents) of geometric measure theory (cf. [M4]) and that it is a real-analytic submanifold on an open dense set.

For the case $m = 1$, Theorem 3.1 classifies all such minimizing curves as simultaneous tracings of circles and straight lines in copies of $\mathbf{R}^2$. Corollary 3.2 infers the general solution to the problem of finding the shortest closed curve of prescribed multi-area. Here Caratheodory's celebrated moment curve ([Ca], [Ga]) makes a surprising appearance in the calculus of variations.

For the case $m = 2$, Proposition 4.2 provides analogous examples but no classification theorem. For $m \geq 3$, the only known examples are unions of constant-mean-curvature hypersurfaces lying in largely orthogonal subspaces (Proposition 4.3).

1.1. The standard thread problem. In contrast to our approach, the standard generalization from $\mathbf{R}^2$ to $\mathbf{R}^n$ of length-minimizing curves C for fixed area and boundary requires a fixed reference curve (or "wire") C_0 with the same boundary and minimizes the length of C given the area of the area-minimizing surface R bounded by $C - C_0$. Actually, in the usual, more physical version of the problem, one fixes the length of C (the "thread") and minimizes the area of R (perhaps a soap film). This problem also generalizes from curves to m-dimensional surfaces. Minimizers generally have constant mean curvature. See [E], [DHKW, Chapt. 10], [N].

1.2. The variational condition. The variational condition 2.2 on our new m-dimensional surfaces S in $\mathbf{R}^n$ minimizing area for given multi-volume and boundary provides an $(m+1)$-vector $\xi \in \Lambda_{m+1}\mathbf{R}^n$ such that the mean curvature vector H satisfies

$$\mathrm{H} = \xi \lfloor \overrightarrow{S},$$

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