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ANALYTIC TORSION AND R-TORSION FOR MANIFOLDS WITH BOUNDARY*

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Abstract. We prove a formula relating the analytic torsion and Reidemeister torsion on manifolds with boundary in the general case when the metric is not necessarily a product near the boundary. The product case has been established by W. Lučk and S. M. Vishik. We find that the extra term that comes in here in the nonproduct case is the transgression of the Euler class in the even dimensional case and a slightly more mysterious term involving the second fundamental form of the boundary and the curvature tensor of the manifold in the odd dimensional case.

1. Introduction. The Reidemeister torsion (R-torsion for short) is a combinatorial/topological invariant associated to a unitary representation of the fundamental group of a manifold. Introduced by Reidemeister [R] and Franz [F], it is used to classify the lens spaces. It was further developed by Milnor and Whitehead and used successfully in classifying the cobordisms. In searching for an analytic interpretation of the R-torsion, Ray and Singer [RS1], [RS2] introduced the analytic torsion, which subsequently found many significant and far-reaching applications (Cf., for example, [B], [BGS], [BL], [GS], [DO], [W]).

The celebrated Cheeger-Müller theorem [C], [Mü1] establishes the Ray-Singer conjecture: namely, on closed manifolds, the analytic torsion is indeed the same as the R-torsion. Further significant work along this line includes that of Müller [Mü2] where he extends the theorem to unimodular representations, that of Bismut-Zhang [BZ], which treats general representations (in which interesting secondary invariants come in), and that of Burghelea-Friedlander-Kappeler-McDonald [BFKM] which deals with infinite dimensional representations (finite type Hilbert module).

All these work deals with closed manifolds. With appropriate boundary conditions, the torsion invariants can also be defined for manifolds with boundary. In fact, the torsions for manifolds with boundary provides an important stepping stone in Cheeger's approach to the Cheeger-Müller theorem (Cf. [C], [RS1]). In [C], the analog of Ray-Singer conjecture on manifolds with boundary is briefly discussed but the geometric information involved is not made explicit. This problem is addressed much later, first in Lott-Rothenberg [LR], and then Lück [L] and Vishik [V]. By assuming a product metric structure near the boundary, they found that the analytic torsion and the R-torsion differ by a topological invariant, namely, (up to a constant), the Euler number of the boundary. This work has been generalized by Burghelea-Friedlander-Kappeler [BFK] to the finite type Hilbert module.

We obtain a general formula of the difference of analyitc torsion and Reidemaster torsion on a Riemannian manifold with boundary, without assuming the product metric structure near the boundary. In this case, there are two extra, geometric terms coming in, involving the second fundamental form of the boundary. In fact, one of the terms is the Chern-Simons class defined by Bismut-Zhang [BZ], transgressing the Euler class. This term therefore vanishes if the dimension of the manifold is odd. The other term, which vanishes for the even dimension, is only slightly more complicated,

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