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ITERATED FUNCTION SYSTEMS WITH OVERLAPS*

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Abstract. In general it is difficult to study the multifractal structure of a self-similar measure when the associated iterated function system does not satisfy the open set condition. In this paper we will give two methods to deal with the overlapping situation. For the first method we make use of a transition matrix to calculate the L^p-scaling spectrum $\tau(p)$ of the measure. The second method depends on the Fourier transformation and the Ruelle operator; we use it to calculate the Sobolev exponent of the measure. We apply these two methods to study the m-th convolution of the Cantor measure, and also an interesting construction investigated recently by Kenyon [K] and Rao and Wen [RW]: $S_0(x) = \frac{1}{3}x,\ S_1(x) = \frac{1}{3}x + \frac{\lambda}{3},\ S_2(x) = \frac{1}{3}x + \frac{2}{3}$, with $0 < \lambda < 1,\ \lambda \in \mathbb{Q}$.

1. Introduction. Let μ be a bounded positive Borel measure on $\mathbb{R}^d$ with compact support. For $p > 0$, the *L^p-scaling spectrum (*or *L^p-scaling exponent)* $\tau(p)$ of μ is defined as

$$\tau(p) = \varliminf_{h\to 0^+} \frac{\ln \sum_i \mu(Q_i(h))^p}{\ln h}, \tag{1.1}$$

where $Q_i(h) = [n_1h,\ (n_1+1)h) \times \cdots \times [n_dh,\ (n_d+1)h),\ (n_1, \ldots, n_d) \in \mathbb{Z}^d$ is an h-mesh cube in $\mathbb{R}^d$ and the sum is taken over all such cubes which intersect the support of μ. (A more general definition of $\tau(p)$ which includes negative values of p can be found in [LN1], [O], [R].) The function $\tau(p)$ plays a central role in the theory of multifractal measures. It is well known that if μ is the self-similar measure defined by a family of contractive similitudes $\{S_j\}_{j=0}^m$ which satisfies the *open set condition* [Hu], then there is an explicit formula to calculate $\tau(p)$ ([CM], [O]). Moreover, the Legendre transformation (concave conjugate) of $\tau(p)$ (i.e., $\tau^*(\alpha) := \inf\{q\alpha - \tau(q) : q \in \mathbb{R}\}$) equals the Hausdorff dimension of the set

$$K(\alpha) = \Big\{x \in \mathrm{supp}(\mu) : \lim_{h\to 0^+} \frac{\ln \mu(B_h(x))}{\ln h} = \alpha\Big\},$$

where $B_h(x)$ denotes the h-ball centered at x, and $\mathrm{supp}(\mu)$ denotes the support of μ. The quantity $\lim_{h\to 0^+} \dfrac{\ln \mu(B_h(x))}{\ln h}$ is known as the *local dimension* of μ at x, the Hausdorff dimension of $K(\alpha)$ as a function of α is called the *dimension* of μ, and the relationship between $\tau(p)$ and the dimension is the well-known *multifractal formalism* (see e.g., [F], [CM]). In general it is difficult to handle $K(\alpha)$ theoretically or computationally. The scaling spectrum $\tau(p)$ and the multifractal formalism hence serve as an important alternative to study the multifractal structure.

Following the terminology of Barnsley [B], we call the above family of contractive similitudes $\{S_j\}_{j=0}^m$ an *iterated function system* (IFS). If the family does not satisfy the open set condition, it is much harder to obtain the scaling exponent $\tau(p)$ and

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