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THE GEOMETRY OF HYPERBOLIC AND ELLIPTIC CR-MANIFOLDS OF CODIMENSION TWO*

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Abstract. The general theory of parabolic geometries is applied to the study of the normal Cartan connections for all hyperbolic and elliptic 6-dimensional CR-manifolds of codimension two. These structures present a very distinguished case in CR-geometry with many features of the non-degenerate real hypersurfaces. The geometric meaning of the individual components of the torsion is explained and the chains of dimensions one and two are discussed.

There have been many attempts to use some ideas going back up to Cartan, in order to understand the geometry of CR-manifolds. In the codimension one cases, the satisfactory solution had been worked out in the seventies, see [22, 23, 8], but the higher codimensions have not been understood yet in a comparable extent. In general, there are no canonical homogeneous models for higher codimensional CR-structures (since there is a continuum of non-isomorphic ones) and, furthermore, the spaces of automorphisms of these homogeneous models are much smaller than in the hypersurface case. There is an exception, however: the 6-dimensional CR-manifolds of codimension two. There are just three non-isomorphic non-degenerate homogeneous models for these geometries and the automorphism groups happen to be semi-simple for two of them. Compared to all other cases, not much has been known about these distinguished geometries, see [9] for more historical remarks and a few results.

In this paper, the recent general theory of the so called parabolic geometries is applied to the latter semi-simple cases. In particular, we use the approach developed in [4, 21], see also [24, 27] for earlier results. Relying on recent achievements by the authors, a clean and quite simple construction of the normal Cartan connection is presented. This Cartan connection replaces the absolute parallelisms from [9] by more powerful geometric tools and it enables the detailed study of geometrical and analytical properties of the CR structures. Consequently the resulting geometric picture is much more transparent and surprising new results are obtained.

The main advantage of our approach is the fully coordinate-free handling of the normal Cartan connection and its curvature. Thus we are able to translate the cohomological properties of the structure algebras into full geometrical understanding of the curvature obstruction, without writing down the curvature components explicitly. The initial section introduces the CR structures and provides a brief exposition of distinguished second order osculations of the surfaces by quadrics. Then we observe, that this osculation transfers enough data from the quadric to apply the general construction of normal Cartan connections, due to [24, 4]. This leads easily to the main Theorems 1.2 and 1.3. In fact, the Cartan connections are constructed also for certain abstract CR-manifolds though the embedded ones have many distinguished properties. The second section is devoted to the exposition of the generalities on parabolic geometries modeled over $|2|$-graded algebras and provides the proof of the existence of the normal connections.

Next we study the local geometry of the hyperbolic points in detail. We recover easily all known facts from [9], but we go much further. In particular, we identify

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