ASIAN J. MATH.
Vol. 4, No. 3, pp. 599–620, September 2000

NEARLY KÄHLERIAN EMBEDDINGS OF SYMPLECTIC MANIFOLDS*

DAVID BORTHWICK† AND ALEJANDRO URIBE‡

1. Introduction and statement of results. Let X, ω be a $2n$-dimensional compact symplectic manifold without boundary. There always exists a compatible almost-complex structure J on X, i.e.

$$\omega(Ju, Jv) = \omega(u, v), \qquad \omega(\cdot, J\cdot) \gg 0.$$

Let g denote the associated Riemannian metric on X, that is $g(u, v) = \omega(u, Jv)$. Assume that ω defines an integral cohomology class, which means that there exists a complex line bundle, $L \to X$, with a connection ∇ whose curvature is $-i\omega$.[1]

Using the notion of "generalized Szegö projectors" [8], one can "quantize" (X, ω, J) by a suitable sequence of finite-dimensional Hilbert spaces, $\mathcal{H}_k \subset C^\infty(X, L^{\otimes k})$, defined for all k sufficiently large. (We'll recall their definition in the next section.) The usual procedure of algebraic geometry yields well-defined maps,

$$F_k : X \longrightarrow \mathbb{P}\mathcal{H}_k^*. \tag{1.1}$$

The dimension of $\mathcal{H}_k$ is the Riemann-Roch polynomial of X evaluated at k, that is

$$\dim \mathcal{H}_k = \int e^{k[\omega]}\tau = \mathrm{Vol}(X)k^n + O(k^{n-1}),$$

where τ is the Todd class.

In this paper we study the asymptotic geometry of the maps F_k. We propose the general philosophy that the quantization $\{\mathcal{H}_k\}$ is a natural object that can be used to study the geometry of (X, ω). The main results are as follows:

THEOREM 1.1. *Let Ω_k and g_k^0 denote respectively the symplectic form and Kähler metric of $\mathbb{P}\mathcal{H}_k^*$. Then there exist positive constants C_1 and C_2 and k_0 such that for all $k > k_0$*

$$\forall x \in X \qquad |\frac{1}{k}F_k^*\Omega_k - \omega|_x \leq \frac{C_1}{k} \tag{1.2}$$

and

$$\forall x \in X \qquad |\frac{1}{k}F_k^*g_k^0 - g|_x \leq \frac{C_2}{k}\,. \tag{1.3}$$

Moreover, F_k is injective for large k and therefore an embedding.

*Received April 7, 1999; accepted for publication May 19, 1999.
†Department of Mathematics and Computer Science, Emory University, Atlanta GA 30322, USA (davidb@mathcs.emory.edu). The author supported in part by NSF grant DMS-9796195 and by an NSF postdoctoral fellowship.
‡Mathematics Department, University of Michigan, Ann Arbor, MI 48109-1109, USA (uribe@math.lsa.umich.edu). The author supported in part by NSF grant DMS-9623054.

[1] This convention is fairly common in complex geometry. But it is also common to include a factor of 2π, and in quantum mechanics there is yet a different convention. The choice affects our normalization of the Fubini-Study form, and will be responsible for certain factors of 2 in Sections 4 and 6.