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HARMONIC MAPS OF DEGREE 1 INTO THE UNIT 2-SPHERE*

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1. Introduction. Let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and M a closed Riemann surface of genus greater than 1. Let $H^1(M, S^2)$ be the set of all $u \in H^1(M, \mathbb{R}^3)$ with $u(x) \in S^2$ for a.e. $x \in M$. The Dirichlet energy functional E on $H^1(M, S^2)$ is given by

$$E(u) = \frac{1}{2}\int_M |\nabla v|^2 dVol_M, \qquad \forall u \in H^1(M, S^2)$$

The critical points of the energy functional are harmonic maps, which satisfy the following Euler-Lagrange equation

$$\triangle u + |\nabla u|^2 u = 0, \tag{1.1}$$

where $\triangle$ is the Laplace-Beltrami operator. Notice that the energy functional does not depend on the choice of the metrics on M, provided that they are compatible with the complex structure on M. So the weak harmonic maps from M to S^2 depend only on the complex structure on M. In this paper, we will consider the following problem: is there a harmonic map of degree 1 from M to S^2? There is definitely no E-minimum, since it would be holomorphic or anti-holomorphic [14]. And it is well known that there is no harmonic map of degree 1 from a torus to a 2-sphere (see [8]). In 1978, using a minimization procedure in a suitable space, L. Lemaire [14] obtained that there exists a harmonic map of degree one from M to S^2 provided that M has three planes of symmetry. Using the result due to J. Jost [13], G.F. Wang [21] improved this result under the condition that M has one plane of symmetry. Here, we will follow a strategy proposed by J.M. Coron [5]; using the Minimax Principle of J. Jost; to prove the existence of such a harmonic map in a different class of Riemann surfaces defined as below. In particular, we do not need any symmetry assumption and our existence result proves the existence of degree 1 harmonic map on all Riemann surfaces described by an open subset of the moduli space of Riemann surfaces. Let R_1 and R_2 be two closed Riemann surfaces with positive genus g_1 and g_2 respectively. For each j, fix a point $p_j \in R_j$, and a coordinate neighborhood (V_j, z_j) around p_j such that $z_j(p_j) = 0$ and $z_j(V_j) = B = \{z_j \in \mathbb{C} : |z_j| < 2\}$. For every complex ϵ_j, with $0 < |\epsilon_j| < 1$, we set

$$V_{j,\epsilon_j} = z_j^{-1}(\{z_j \in \mathbb{C} : |\epsilon_j| \le |z_j| \le 1\}), \qquad j = 1, 2.$$

Now, fix two points q_1 and $q_2 \in S^2$, and two coordinate neighborhoods (U'_j, z'_j) around q_j such that $z'_j(q_j) = 0$ and $z'_j(U'_j) = B(0,1) = \{z'_j \in \mathbb{C} : |z'_j| < 1\}$. We define

$$U'_{j,\epsilon_j} = U'_j - (z'_j)^{-1}(\{z'_j \in \mathbb{C} : |z'_j| \le |\epsilon_j|\}), \qquad j = 1, 2.$$

Then, identifying U'_{j,ϵ_j} and V_{j,ϵ_j} for $j = 1, 2$ by the mappings

$$z'_j \cdot z_j = \epsilon_j,$$

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