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EXISTENCE AND UNIQUENESS OF MINIMAL GRAPHS IN HYPERBOLIC SPACE*

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Abstract. In this paper we prove general existence and uniqueness theorems for minimal vertical graphs in hyperbolic space.

Introduction. In this paper we shall prove general existence and uniqueness results for vertical minimal graphs in the hyperbolic space $\mathbf{H}^3 = \{(x_1, x_2, x_3) \in \mathbf{R}^3,\ x_3 > 0\}$ over bounded and unbounded domains in $\partial_\infty \mathbf{H}^3$ (see Eq. 1-1). We solve the Dirichlet problem for the minimal equation (Problem (P) in §1) in two variables x_1, x_2 over bounded C^0 convex domains, given arbitrary non negative continuous boundary value data (see Theorem 2.3). If the boundary data and the domain are smooth enough an analogous result also holds in every dimension (See Theorem 2.2).

The main result of this paper is the derivation of the *Perron process* for the minimal vertical equation in the upper halfspace model of hyperbolic three space (see Problem (P) in §1). More precisely, we shall infer that existence of a *supersolution* yields a *solution* for the Dirichlet Problem (P). When the boundary $\partial\Omega$ is C^0 convex at p, or the boundary data f vanish at p, and p has a *barrier* (see Definition 1.6) then the solution given by Perron process is continuous up to p and takes the given boundary data at p. In particular, in any convex arc $\Gamma \subset \partial\Omega$ such a solution is continuous up to Γ. Moreover, if p has a barrier and $f \geq \alpha > 0$ then the solution given by Perron process is continuous up to the boundary and takes the prescribed boundary value f at p (see Theorem 3.4). We shall infer by Perron process, several *existence* and *uniqueness* results for minimal *complete* and *non complete* graphs over C^0 *unbounded convex* and *non convex* domains, namely complete minimal graphs, invariant by a given 1-parameter group of hyperbolic or parabolic isometries of $\mathbf{H}^3$ (see Propositions 3.9 and 3.12). We shall also infer from Perron process the existence of the Dirichlet problem for the minimal vertical equation over a band, given arbitrary non negative continuous boundary data (see Theorem 4.1). Theoretically, this result can be viewed as an example of application of *Perron process.* For a convex unbounded domain in a band Ω, we shall prove the following uniqueness result: if the prescribed boundary data f is bounded, non negative and uniformly continuous then the Dirichlet problem for the minimal vertical equation admits at most one solution. Moreover, any such a solution is bounded and uniformly continuous. In fact this result holds in higher dimensions (see Corollary 1.11 and Remark 1.9). From Perron process we shall deduce a maximum principle for domains in a proper sector (see Theorem 4.3) that gives rise to an existence result for the Dirichlet problem (P) over convex domains in a proper sector (see Theorem 4.5 and Remark 4.7). Next, we explain the main tools.

Throughout this paper, we use L. Simon interior estimates for mean curvature type equations in two variables as a major tool, see [Si], together with the control at

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