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# NÉRON MODELS FOR SEMIABELIAN VARIETIES: CONGRUENCE AND CHANGE OF BASE FIELD*

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**1. Introduction.** Let $\mathcal{O}$ be a henselian discrete valuation ring with perfect residue field. Denote by $K$ the fraction field of $\mathcal{O} = \mathcal{O}_K$, and by $\mathfrak{p} = \mathfrak{p}_K$ the maximal ideal of $\mathcal{O}$. Then every abelian variety $A$ over $K$ has a Néron model $\underline{A}^{\mathrm{NR}}$ over $\mathcal{O}$. The Néron model $\underline{A}^{\mathrm{NR}}$ of $A$ is a smooth group scheme of finite type over $\mathcal{O}$, characterized by the property that for every finite unramified extension $L$ of $K$, every $L$-valued point of $A_K$ extends uniquely to an $\mathcal{O}_L$-valued point of $\underline{A}^{\mathrm{NR}}$. We refer to the book [BLR] for a thorough exposition of the construction and basic properties of Néron models.

In general, the formation of Néron models does not commute with base change. Rather, for every finite extension field $M$ of $K$, we have a canonical homomorphism

$$\mathrm{can}_{A,M/K} : \underline{A}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_M \to \underline{A_M}^{\mathrm{NR}}$$

from the base change of the Néron model to the Néron model of the base change, which extends the natural isomorphism between the generic fibers. If $A$ has semistable reduction over $\mathcal{O}_K$, i.e. if the neutral component of $\underline{A}^{\mathrm{NR}}$ is a semiabelian scheme, then $\mathrm{can}_{A,M/K}$ is an open immersion. We define a numerical invariant $c(A,K)$ of $A$ as follows. Let $L/K$ be a finite separable extension such that the abelian variety $A$ has semistable reduction over $\mathcal{O}_L$. Let $e(L/K)$ be the ramification index of $L/K$. Define

$$c(A,K) := \frac{1}{e(L/K)} \mathrm{length}_{\mathcal{O}_L} \left( \frac{\mathrm{Lie}\, \underline{A_L}^{\mathrm{NR}}}{\mathrm{can}_{A,L/K\,*}(\mathrm{Lie}\, \underline{A}^{\mathrm{NR}} \otimes \mathcal{O}_L)} \right)$$

Notice that $c(A,K)$ does not depend on the choice of $L/K$. The invariant $c(A,K)$ measures the failure of $A$ to have semistable reduction over $\mathcal{O}_K$; it is equal to zero if and only if $A$ has semistable reduction over $\mathcal{O}_K$. Thus $c(A,K)$ may be regarded as a sort of "conductor" of $A$. We will call it the *base change conductor* of $A$. One motivation of this paper is to study the properties of the invariant $c(A,K)$ and determine whether it can be expressed in terms of more familiar ones, for instance the Artin conductor or the Swan conductor of the $\ell$-adic Tate module attached to $A$.

In a similar fashion one can attach to each semiabelian variety $G$ over $K$ a non-negative rational number $c(G,K)$; see 2.4. for the precise definition. For a torus $T$ over $K$ it has been shown, by E. de Shalit and independently by J.-K. Yu and the author, that

$$(*) \qquad c(T,K) = \frac{1}{2}\, a\left(X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}\right) ,$$

one-half of the Artin conductor of the linear representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the character group of $T$; see [CYdS]. This answers, in the affirmative, a question posed by B. Gross and G. Prasad.

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