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# ON WARING'S PROBLEM FOR THREE SQUARES AND AN $\ell$TH POWER*

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**1. Introduction.** Professor Loo Keng Hua was probably most renowned for his important contributions to the theory of Waring's problem. It is therefore fitting that this dedicatory article should be devoted to the proof of the hitherto elusive asymptotic formula for the number $\nu(n)$of representations of a large number $n$ as the sum of three squares of integers of either sign and a non-negative (non-linear) $\ell$th power. Of the form that the Hardy-Littlewood circle method would presage, our asymptotic formula does not, however, seem to fall within the current ambit of that method for $\ell > 2$ and is therefore treated by other means that include the theories of Dirichlet's $L$-functions with real characters and the representation of numbers as the sum of three squares. All the same, as the proof develops, we encounter some of the familiar ancillary apparatus of the circle method, for details of which we most aptly direct the reader to Hua's compendium on number theory ([7], to which we refer later as H).

The situation under examination is one of those in additive number theory where the establishment of an asymptotic formula for the number of representations of an integer in a proposed form lies much deeper than any consequential criterion for deciding which integers possess such a representation. The latter question, indeed, has been settled by Churchhouse [2] in this case and does not cause undue difficulty owing to the familiar simple condition for the representability of a number as a sum of three squares. Nevertheless, conditions of the Churchhouse type emerge so naturally in the course of our proof that they are embodied explicitly in our final theorems.

We should mention that this article represents the final phase of work that was announced in our address to the I. C. M. Congress of 1983 in Warsaw [6]. Our earlier researches had, indeed, advanced the situation from Jacobi's exact formula for the case $\ell = 2$ (see H, p 216) to asymptotic formulae for exponents as large as $\ell = 7$ but had until then failed to deal with the general case.

**2. Notation.** Although the meaning of most of the notation is self evident from the content, the following guide may be helpful. The letters $n$ and $n_1$ are usually positive integers and will be regarded as tending to infinity in the later parts of the work; $X_i, X_i'$ are integers; $\varepsilon$ is an arbitrarily small number that is not necessarily the same at each occurrence; the given exponent $\ell$ is assumed to exceed 1; $B_i$ is a positive constant that depends at most on $\ell; B_i(\varepsilon)$ is a positive constant depending at most on $\ell$ and $\varepsilon$; the constants implied by the $O$-notation are of type $B_i$ save when there is an $\varepsilon$ occurring as exponent, in which case they are of type $B_i(\varepsilon)$.

The highest common factor of integers $X_1, X_2, X_3, X_4$ is $(X_1, ..., X_4)$ when it is defined; $\sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha}$ and $d(n) = \sigma_0(n)$.

For odd $k$, $\left(\frac{m}{k}\right) = (m|k)$ is the Jacobi symbol of quadratic residuacity.

**3. Formulae regarding the representation of a number as a sum of**

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