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SPIN^C MANIFOLDS AND RIGIDITY THEOREMS IN K-THEORY*

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Abstract. We extend our family rigidity and vanishing theorems in [LiuMaZ] to the Spin^c case. In particular, we prove a K-theory version of the main results of [H], [Liu1, Theorem B] for a family of almost complex manifolds.

0. Introduction. Let M, B be two compact smooth manifolds, and $\pi : M \to B$ be a smooth fibration with compact fibre X. Let TX be the relative tangent bundle. Assume that a compact Lie group G acts fiberwise on M, that is, the action preserves each fiber of π. Let P be a family of G-equivariant elliptic operators along the fiber X. Then the family index of P, $\mathrm{Ind}(P)$, is a well-defined element in $K(B)$ (cf. [AS]) and is a virtual G-representation (cf. [LiuMa1]). We denote by $(\mathrm{Ind}(P))^G \in K(B)$ the G-invariant part of $\mathrm{Ind}(P)$.

A family of elliptic operator P is said to be *rigid on the equivariant Chern character level* with respect to this G-action, if the equivariant Chern character $\mathrm{ch}_g(\mathrm{Ind}(P)) \in H^*(B)$ is independent of $g \in G$. If $\mathrm{ch}_g(\mathrm{Ind}(P))$ is identically zero for any g, then we say P has *vanishing property on the equivariant Chern character level.* More generally, we say that P is *rigid on the equivariant K-theory level*, if $\mathrm{Ind}(P) = (\mathrm{Ind}(P))^G$. If this index is identically zero in $K_G(B)$, then we say that P has *vanishing property on the equivariant K-theory level.* To study rigidity and vanishing, we only need to restrict to the case where $G = S^1$. From now on we assume $G = S^1$.

As was remarked in [LiuMaZ], the rigidity and vanishing properties on the K-theory level are more subtle than that on the Chern character level. The reason is that the Chern character can kill the torsion elements involved in the index bundle.

In [LiuMaZ], we proved several rigidity and vanishing theorems on the equivariant K-theory level for elliptic genera. In this paper, we apply the method in [LiuMaZ] to prove rigidity and vanishing theorems on the equivariant K-theory level for Spin^c manifolds, as well as for almost complex manifolds. To prove the main results of this paper, to be stated in Section 2.1, we will introduce some shift operators on certain vector bundles over the fixed point set of the circle action, and compare the index bundles after the shift operation. Then we get a recursive relation of these index bundles which will in turn lead us to the final result (cf. [LiuMaZ]).

Let us state some of our main results in this paper more explicitly. As was remarked in [LiuMaZ], our method is inspired by the ideas of Taubes [T] and Bott-Taubes [BT].

For a complex (resp. real) vector bundle E over M, let

$$\begin{aligned} \mathrm{Sym}_t(E) &= 1 + tE + t^2 \mathrm{Sym}^2 E + \cdots, \\ \Lambda_t(E) &= 1 + tE + t^2 \Lambda^2 E + \cdots \end{aligned} \tag{0.1}$$

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