ASIAN J. MATH.
Vol. 4, No. 4, pp. 977–986, December 2000

014

ON THE DISTRIBUTION OF CERTAIN HUA SUMS*

S. J. PATTERSON†

1. Introduction. Let $f(x)$ be a polynomial with integral coefficients. Let, for $c \geq 1$

$$S\big(f(x), c\big) = \sum_{j=0}^{c-1} e^{2\pi i f(j)/c}$$

be the corresponding Hua sum. One has the estimate, due to Hua,

$$S\big(f(x), c\big) = O(c^{1-\frac{1}{n}})$$

where $n = \deg(f)$. In this paper we shall study the asymptotic behaviour of

$$\sum_{c \leq X} S\big(f(x), c\big)$$

in the special case of $f(x) = a\ x^3$. In general there is some numerical evidence to suggest that

$$\sum_{c \leq X} S\big(f(x), c\big) \sim K(f) \cdot X^{1+\frac{1}{n}}$$

unless f is symmetric in the sense that there exists $r \in \mathbb{Z}$ so that $f(r - x) = f(x)$. If this is so, and it can only happen if n is even, there is a little evidence to suggest that

$$\sum_{c \leq X} S\big(f(x), c\big) \sim K'(f) \cdot X^{1+\frac{2}{n}}$$

when $n > 2$. I shall discuss these problems in a paper which is, at present, in preparation.

In the case of $\deg(f) = 3$ then it is possible to prove results of this type for a certain class of f over $\mathbb{Z}[\omega](\omega^2 + \omega + 1 = 0)$. This is the subject of a forthcoming paper by R. Livné and the author ([LP]). The method used there is based on the theory of cubic metaplectic forms and an identity due to W. Duke and H. Iwanieć, [DI]. Unexpectedly for certain very special f, namely those of the form $a_3 \cdot x^3$ one has, at least for some a_3

$$\sum_{\substack{c:N(c) \leq X \\ c \equiv 1 \pmod 3}} \widetilde{S}(a_3 x^3, c) \sim k(a_3) \cdot X^{\frac{4}{3}} \log X$$

where, here, $\widetilde{S}$ ist the analogue of S for $\mathbb{Z}[\omega]$ and $c \in \mathbb{Z}[\omega]$. The purpose of this paper is to use the theory of metaplectic forms again to prove the following:

*Received June 29, 2000; accepted for publication October 4, 2000.
†Department of Mathematics, Georg-August-University, Bunsenstrasse 3-5, D-37073 Goettingen, Germany (sjp@uni-math.gwdg.de).