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BOUNDS FOR CERTAIN EXPONENTIAL SUMS*

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1. Introduction. In this paper we consider exponential sums of the type

$$S(ax^n + bx, p^m) = \sum_{x \bmod p^m} e(ax^n + bx/p^m), \tag{1.1}$$

and

$$S(ax^n + bx, \chi, p^m) = \sum_{x \bmod p^m} \chi(x) e(ax^n + bx/p^m), \tag{1.2}$$

where p^m is a prime power , $\chi \bmod p^m$ is a Dirichlet character, a, b, n are integers with $n \geq 2$.

The first sum was studied in connection with Waring's problem and we have a classical result due to professor Hua [10]. The second sum has not been studied before as far as the authors know. We hope it can be used in the work of generalizing Waring's problem. In [6], Davenport and Heilbronn showed that

$$S(ax^n + bx, p^m) \ll_n p^{\theta m}(b, p^m), \quad \text{if } p \nmid a, \tag{1.3}$$

where $\theta = 2/3$ if $n = 3$, and $\theta = 3/4$ if $n \geq 3$. Hua [10] showed that $\theta = 1/2$ for all $n \geq 2$ (also see Lemma 4.1 of Vaughan [19]). Hua's proof depends on Weil's estimate for exponential sums over finite fields (see Bombieri [1] or Schmidt [15]). In this case we have

$$|S(ax^n + bx, p)| \leq (n-1)p^{\frac{1}{2}}, \quad \text{if } p \nmid a. \tag{1.4}$$

For $m \geq 2$, following the work of Loxton and Smith [13], and Smith [17], one has that

$$|S(ax^n + bx, p^m)| \leq d_{n-1}(p^m) p^{\frac{m}{2}} (\Delta, p^m)^{\frac{1}{2}}, \tag{1.5}$$

where $d_{n-1}(p^m)$ is the number of representations of p^m as a product of $n-1$ positive integers and Δ is the discriminant of the derivative of the polynomial $ax^n + bx$. After an improvement by Loxton and Vaughan, Theorem 1 of [14] implies the following estimate

$$|S(ax^n + bx, p^m)| \leq (n-1)p^{\frac{m+\tau_0}{2}} (D, p^m)^{\frac{1}{2}}, \tag{1.6}$$

where D is the different of the derivative of the polynomial $ax^n + bx$, and $\tau_0 = 1$ if $p \leq n$, and $\tau_0 = 0$ if $p > n$. Very recently, Dabrowski and Fisher established better bounds for exponential sums of this kind. Under the restriction $p \nmid ab$, $p \nmid n$, and

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