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REAL EMBEDDINGS AND THE ATIYAH-PATODI-SINGER INDEX THEOREM FOR DIRAC OPERATORS*

XIANZHE DAI† AND WEIPING ZHANG‡

Abstract. We present the details of our embedding proof, which was announced in [DZ1], of the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary [APS1].

Introduction. The index theorem of Atiyah, Patodi and Singer [APS1, (4.3)] for Dirac operators on manifolds with boundary has played important roles in various problems in geometry, topology as well as mathematical physics. Not surprisingly then, there are by now quite a number of proofs of this index theorem other than Atiyah, Patodi and Singer's original proof [APS1]. Among these proofs we mention those of Cheeger [C1, 2] (see also Chou [Ch]), Bismut-Cheeger [BC1] and Melrose [M]. One common point underlying all these proofs (including the original one) is that they can all be viewed, in one way or another, as certain extensions to manifolds with boundary of the heat kernel proof of the local index theorem for Dirac operators on closed manifolds (cf. [BeGV]). That is, one starts with a Mckean-Singer type formula and then studies the small time asymptotics of the corresponding heat kernels. In particular, one makes use of the explicit formulas for the heat kernel of the Laplace operators on the cylinder ([APS1], [M]) and/or cone ([BC1], [C1, 2], [Ch]) (being attached the boundary) for the analysis near the boundary. The η-invariant on the boundary, which was first defined in [APS1], appears naturally during the process.

Now recall that Atiyah and Singer [AS] also have a K-theoretic proof of their index theorem for elliptic operators on closed manifolds. In such a proof, one transforms the problem, through direct image constructions in K-theory, to a sphere and then applies the Bott periodicity theorem on the sphere to establish the result.[1] It is thus natural to ask whether the strategy of Atiyah-Singer's K-theoretic ideas can be used to prove the Atiyah-Patodi-Singer index theorem for manifolds with boundary. The purpose of this paper is to present such a proof, of which an announcement of basic ideas has already appeared in [DZ1].

Briefly speaking, we embed the manifold with boundary under consideration into a ball, instead of a sphere, so that it maps the boundary of the original manifold to the boundary sphere of the ball, and reduce the problem to the ball. Now since any vector bundle on the ball is topologically trivial, one obtains the result immediately. This works even when the original manifold has no boundary, giving a proof of the Atiyah-Singer index theorem for Dirac operators. The Bott periodicity theorem is thus not needed.

Observe that in [AS], Atiyah and Singer made heavy use of the techniques of pseudodifferential operators, which is not suitable for treating directly the global elliptic boundary problems. This is the first serious difficulty in extending directly the arguments in [AS] to deal with the Atiyah-Patodi-Singer boundary problems.

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†Department of Mathematics, University of California, Santa Barbara, California 93106, USA (dai@math.ucsb.edu). Partially supported by NSF and Alfred P. Sloan Foundation.
‡Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China (weiping@sun.nankai.edu.cn). Partially supported by the CNNSF, EMC and the Qiu Shi Foundation.
[1] See also the book of Lawson-Michelsohn [LM] for a comprehensive treatment of this approach.