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EXISTENCE OF EXTREMAL METRICS ON ALMOST HOMOGENEOUS MANIFOLDS OF COHOMOGENEITY ONE*

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Dedicated to Professor L. K. Hua, who was the teacher of Professor J. Q. Zhong. Zhong was the teacher of Guan in China and led him into this topic. Guan still have copies of papers of Calabi and Futaki which Zhong showed him in 1986.

1. Introduction. It is well-known that every Kähler class of a compact Kähler homogeneous manifold admits a Kähler metric with constant scalar curvature. We may ask following question:

Can we find any special metric on a compact almost homogeneous manifold of cohomogeneity One?

We have found this kind of metrics for a compact almost homogeneous manifold with two ends in [Gu1,2,3], i.e., there are both extremal metrics and quasi-Einstein metrics in each Kähler class[1]. Then we may ask if the same is true for all the compact almost homogeneous manifolds of cohomogeneity one, which are described in both [Ah] and [HS].

If the algebraic part of the automorphism group is not semisimple, the manifold has a structure of a completion of a $\mathbf{C}^*$ bundle over a compact homogeneous Kähler manifold (see Theorem 1). According to the results in [Gu1,2,3], there always exists an extremal metric (resp. a quasi-Einstein metric) in each Kähler class. Hence, the problem of finding extremal metrics (resp. quasi-Einstein metrics) is reduced to the situation in which the algebraic part of the automorphism group is semisimple. By abusing the language, we call this case the *semisimple case*.

The semisimple case is also very interesting since any invariant Kähler metric is not a *very good* metric in the real Riemannian geometry. It is not a metric invariant under a maximal compact Lie group. For example, if a general hypersurface orbit K/L has a right circle action S_1 (there is one always in our case), then the whole manifold has a circle action introduced by this action. By averaging method, we obtain a Riemannian metric which is invariant under $K \times S_1$. However, the new metric can not be Kähler since otherwise the automorphism group of M is bigger than $K_{\mathbf{C}} \times \mathbf{C}^*$—a contradiction to our assumption. So *some methods in the real Riemannian manifold of cohomogeneity one, e.g., the method of circle action, do not work for any one here.*

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[1] In our preparing of [Gu6], we find that the proof of the existence for Kähler metrics of constant scalar curvature on almost homogeneous manifolds with two ends and reductive automorphism group in [Hw1] is not complete. For example, the Theorem 3 there follows from his Theorem 2, but in the case of Theorem 2 the base manifold is a product of manifolds with $b_2 = 1$. Therefore, the proof of his Theorem 4 can come from neither his Theorem 3 nor his Theorem 2. Neither the proof of existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one is true—as we see in this paper. Also, there should be nonnegative conditions of the Ricci curvatures in the Theorem 2 and Corollary 2 in [Gu3], the "B" in the (3) of [Gu3 p.374] should be "B and r". Although the proof there actually works for the case in which the traces r^j of r on the eigenspaces of B are constants with at most one negative r^j which corresponds to an eigenvalue b_j of B such that there is no other eigenvalue of B between b_j and zero, e.g., $B = kg_t$, g_t have constant scalar curvatures.