ASIAN J. MATH.
Vol. 4, No. 4, pp. 831–838, December 2000

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A SIEVE APPLICATION*

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The aim of this note is to prove the following

THEOREM. *Suppose $g \geq 3$ is fixed. There exist positive absolute constants A and B, and a g-term arithmetic progression in the interval $(1, 1+g^{Ag})$, so that the prime factors of all the terms of this progression are at least as large as g^B.*

Several years ago Professor W. M. Schmidt asked me for a result of this kind (he found later that he could prove what he needed in another way), and at the end of his plenary lecture at the 1997 Schinzel Conference in Zakopane, Poland, Professor H.-P. Schlickewei raised a similar question. I thought it might be of some interest to show how the question could be formulated as a sieve problem, and answered in explicit form by means of a simple sieve method.

We formulate the stated result as follows: Let

$$F(n) := (1+n)(1+2n)\dots(1+gn)$$

and

$$\mathcal{A} := \{F(n) : n \in \mathbb{N}, 1 \leq n \leq x\}.$$

Next, let $\mathcal{P}$ be the set of all primes p less than a number $z > 2$, and introduce the counting number

$$S(\mathcal{A}, \mathcal{P}) = |\{a \in \mathcal{A} : (a, \prod_{p<z} p) = 1\}|.$$

If there exist x and z such that

$$S(\mathcal{A}, \mathcal{P}) > 0, \tag{1}$$

this means that there does exist a g-term arithmetic progression

$$1 + rn \ (r = 1, 2, \dots, g)$$

with common difference n satisfying $1 \leq n \leq x$, that lies in the interval

$$(1, 1+gx]$$

and has all its g terms made up of primes at least as large as z. To prove the theorem it is enough to show that there exist constants A and B such that (1) holds with

$$x = g^{Ag-1}, \quad z = g^B. \tag{2}$$

We set about proving (1), with x and z as given in (2), by means of the Brun-Hooley sieve (see [HO]) as presented in [FH]; this method is especially easy to apply, but it should be said that any version of a fundamental lemma (see, for example,

*Received January 6, 2000; accepted for publication March 29, 2000.
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