ASIAN J. MATH.
Vol. 4, No. 4, pp. 839–854, December 2000

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ON THE UNIFORM EQUIDISTRIBUTION OF LONG CLOSED HOROCYCLES*

DENNIS A. HEJHAL†

1. Introduction. I first met Hua during the 1979 Analytic Number Theory Symposium in Durham, England. Hua was quite stimulated by the talk I gave there on pseudo cusp forms for $PSL(2,\mathbb{Z})$ [5] and, for several years thereafter, was a major source of encouragement to me as I began working more systematically with computational spectral theory on Fuchsian groups.

I retain many warm memories from the discussions we had during this period.

Though the spectral techniques in the present paper are decidedly *non*-computational, it is a curious fact that the issue considered here first arose while trying to place certain machine-based heuristics, specifically those of [11], on a more satisfactory geometric footing.

Paper [11] is part of the computational series [6-12] that was first envisioned in conversations with Hua.

In light of this, it seems fair to regard the present paper's main result, theorem A in §3, as having a genesis which springs partially at least from Hua's enthusiasm for computationally-oriented mathematics.

2. Some preliminaries. To get started, we need a bit of notation. Let Γ be any cofinite Fuchsian group acting on the Poincaré upper half-plane H. Assume that $\Gamma \setminus H$ has only one cusp. By making an auxiliary conjugation of Γ, one can position this cusp at $i\infty$ and arrange things so that the isotropy subgroup Γ_∞ is generated by the translation $S(z) = z + 1$. At the same time, it is convenient to equip Γ with a standard fundamental polygon $\mathcal{F}$ which contains the half-strip $[0,1] \times [B,\infty)$ for some $B \geqq 1$. (Cf. [4, pp. 3-5] and [15, pp. 59,61].)

The numerical set-up of both [11] and [12] necessitates looking at the pull-back of $\{0 \leqq x \leqq 1, y = h\}$ inside $\mathcal{F}$ for small values of h. We'll denote this pull-back by $\mathcal{C}_h$.

Since $ds = |dz|/y$, the locus $\mathcal{C}_h$ has hyperbolic length $1/h$. On the other hand, since $S \in \Gamma$, its projection $\pi(\mathcal{C}_h)$ on $\Gamma \setminus H$ is manifestly a closed curve; one calls $\pi(\mathcal{C}_h)$ a *closed horocycle*. (Note that $\pi(\mathcal{C}_h)$ is real-analytic.)

In view of the fact that $1/h \to \infty$ as $h \to 0$, it is only natural to wonder where $\pi(\mathcal{C}_h)$ "goes".

Computer tests show some beautiful patterns and quickly suggest that $\mathcal{C}_h$ becomes everywhere dense in $\mathcal{F}$ as $h \to 0$. Much more, in fact, is true. The set $\mathcal{C}_h$ actually becomes equidistributed with respect to Poincaré area μ as the parameter h decays. (Here $d\mu = y^{-2}dxdy$.) By elementary functional analysis, this is the same thing as saying that

$$\int_0^1 f(x+ih)dx \longrightarrow \frac{1}{\mu(\mathcal{F})}\int_{\mathcal{F}} f(z)d\mu(z) \tag{2.1}$$

*Received August 7, 2000; accepted for publication October 7, 2000.

†Department of Mathematics, Uppsala University, S-75106 Uppsala, Sweden; and School of Mathematics, University of Minnesota, Minneapolis, Mn. 55455, USA (hejhal@math.uu.se, hejhal@math.umn.edu). While preparing this paper, the author was supported by the Göran Gustafsson Foundation for Research in Natural Science and Medicine, and by grants from the Institute for Advanced Study and NEC Research Institute, Inc.