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REGULARITY OF $\overline{\partial}$ ON PSEUDOCONCAVE COMPACTS AND APPLICATIONS*

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Abstract. We prove the regularity of the $\overline{\partial}$-equation for $\mathcal{O}(m)$-valued (p,q)-forms on pseudoconcave compacts of the complex projective space $\mathbb{CP}_n$. This leads to the vanishing of the cohomology groups of the $(0,q)$-forms with coefficients in the Sobolev space $W^k(\Omega)$, $q \neq n-1$, where Ω is a pseudoconcave domain with Lipschitz (respectively C^2) boundary of $\mathbb{CP}_n$ for $k \geq 2$ (respectively $k \geq 1$). As an application, we show that the holomorphic functions of $W^1(\Omega)$ vanish on pseudoconcave domains Ω with C^2 boundary of $\mathbb{CP}_n$. This implies the Hartogs-Bochner phenomenon for CR functions in the space $W^{1/2}(\partial\Omega)$, where Ω is a domain with Lipschitz boundary of $\mathbb{CP}_n$ such that the complement is connected and contains a pseudoconcave domain with C^2 boundary. We also obtain the dual of the k-weighted Bergman space of a pseudoconvex domain in $\mathbb{CP}_n$ as the cohomology group of the $(n,n-1)$-forms with W^k-coefficients on the complement.

1. Introduction. The classical Hartogs-Bochner theorem [4] states that a CR function f on the boundary of a bounded domain $\Omega \subset \mathbb{C}^n$ $(n \geq 2)$ with smooth connected boundary has a holomorphic extension to Ω.

A proof of this theorem may be obtained by using the jump formula for the Bochner-Martinelli transform: $f = f_+|_{\partial\Omega} - f_-|_{\partial\Omega}$, where f_+ is holomorphic on Ω and f_- is holomorphic on $\mathbb{CP}_n \backslash \overline{\Omega}$, where $\mathbb{CP}_n$ is the n dimensional complex projective space. Since $\mathbb{CP}_n \backslash \Omega$ is connected and contains complex lines, it follows that $f_- \equiv$ *constant*. An essential fact for this proof of the Hartogs-Bochner phenomenon is that $\mathbb{CP}_n \backslash \Omega$ contains a pseudoconcave domain.

It is well-known that the Hartogs-Bochner theorem is true for relatively compact domains in a Stein manifold X ($\dim_{\mathbb{C}} X \geq 2$). Important versions of this theorem were obtained by Fichera [11], Kohn and Rossi [25], Grauert and Riemenschneider [13], Harvey and Lawson [18]. Recently, interesting new facts about this phenomenon were obtained by Napier and Ramachandran [29], Dingoyan [8] and Sarkis [33].

In this paper we obtain the following new version of the Hartogs-Bochner theorem in $\mathbb{CP}_n$, $(n \geq 2)$:

(1) Let Ω be a domain with Lipschitz boundary in $\mathbb{CP}_n$ such that $\mathbb{CP}_n \backslash \Omega$ is connected and contains a pseudoconcave domain with C^2 boundary. Then every CR Sobolev $W^{1/2}$-function on the boundary has a holomorphic extension to Ω.

One of the difficulties to prove this result is that there exist pseudoconvex domains Ω in $\mathbb{CP}_2$, such that $\overline{\Omega}$ has no any Stein neighborhood. So we can not use the above mentioned results. Another difficulty is that there exists pseudoconcave domains in $\mathbb{CP}_n$ which do not contain any algebraic hypersurface of $\mathbb{CP}_n$ [10].

It is interesting to mention that the Hartogs-Bochner phenomenon is also valid in distribution categories for bounded domains in $\mathbb{C}^n$ with rectifiable boundary. However, we give an example of pseudoconvex domains in $\mathbb{CP}_2$ such that the Hartogs-Bochner phenomenon does not work for $W^{-1/2}$-functions on the boundary.

The jump formula for the Bochner-Martinelli transform in $\mathbb{CP}_n$ [20] shows that (1) is equivalent to the following version of Liouville's theorem:

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