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REAL HYPERSURFACES IN KÄHLER MANIFOLDS*

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0. Introduction. The present work is concerned with the geometry of embedded real hypersurfaces in a Kähler manifold, where isomorphisms are both holomorphic and isometric in the underlying Riemannian structure. Here we introduce their local invariants and compatibility relations, solve the local realization problem, and give a characterization of the metric sphere in $\mathbb{C}^n$ via a maximum principle for an adapted Laplace operator. A more detailed description follows.

a) In 1.3 we define a doubly covariant tensor ℓ on the complexified tangent bundle of the hypersurface N. It describes the curvature of the maximal complex subbundle HN of TN in the ambient manifold with regard to the Kähler metric . One part of ℓ is the well-known Levi form of the hypersurface with respect to a one-form of norm one that annihilates the "horizontal" tangent bundle HN. The form ℓ may be expressed in terms of the second fundamental form of the hypersurface in an appropriate frame of the underlying Riemannian manifold, see 1.4.

b) In 2 we derive differential compatibility conditions of covariant derivatives of ℓ. In 2.3, 2.4 and 2.5 we employ a connection on the horizontal bundle that is the projection of the Levi-Civita connection of the ambient manifold. It is a one-form on the hypersurface with values in the Lie algebra of the unitary group of the horizontal subbundle and we give its structure equations.

c) In 3.2 we solve the local existence problem as follows. The data are a CR-manifold U with a metric on TU that is hermitian on HU and a real function r that prescribes the ambient curvature of the curves in N that are orthogonal to HN. We require that the composition of the defining form for HN with the complex structure is closed, the complex structure on the horizontal bundle is parallel with respect to the associated Levi-Civita connection, and that an associated unitary connection has vanishing curvature. The latter corresponds to Gauss and Codazzi compatibility conditions. The conclusion is that there exists a local isometric CR-embedding of U into $\mathbb{C}^n$. In 3.4 we embed nondegenerate CR manifolds U together with a prescribed hermitian metric on HU rather than on TU as in 3.2. In exchange we prescribe additional data, namely mixed horizontal and vertical coefficients of ℓ. In this case, there is an embedding under analogous requirements as before. This result has features of both the classical hypersurface existence theorem in Riemannian geometry and of Kuranishi's CR embedding [Ku]. Namely we require compatibility conditions involving metric and complex structure, and the non-degeneracy of the Levi-form.

d) In 4 we give commutator identities for the operation of the Laplace operator of HN on ℓ. In the Riemannian case these are referred to as Simons identity, see [S] and [CdCK].

e) As an application we give in 5 the following

THEOREM 5.2. *Let N be a compact strictly pseudoconvex hypersurface of $\mathbb{C}^n$. Assume that the horizontal mean curvature of N is constant, and $H^{1,0}N$ is parallel in $T^{1,0}\mathbb{C}^n$. Then N is a metric sphere.*

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