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011

CHARACTERISTIC CLASSES FOR $GO(2N, \mathbb{C})$*

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The complex Lie group $GO(n)$ is by definition the closed subgroup of $GL(n)$ consisting of all matrices g such that ${}^tgg$ is a scalar matrix λI for some $\lambda \in \mathbb{C}^*$. (We write simply $GL(n)$, $SO(n)$, $O(n)$, etc. for the complex Lie groups $GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $O(n, \mathbb{C})$, etc.)

The group $GO(1)$ is just $\mathbb{C}^*$, with classifying space $B\mathbb{C}^* = \mathbf{P}^\infty_{\mathbb{C}}$, whose cohomology ring $H^*(B\mathbb{C}^*; \mathbb{Z}/(2))$ is the polynomial ring $\mathbb{Z}/(2)[\lambda]$ where $\lambda \in H^2(B\mathbb{C}^*; \mathbb{Z}/(2))$ is the Euler class. For an odd number $2n+1 \geq 3$, the group $GO(2n+1)$ is isomorphic to the direct product $\mathbb{C}^* \times SO(2n+1)$. Hence $BGO(2n+1)$ is homotopic to the direct product $B\mathbb{C}^* \times BSO(2n+1)$, with cohomology ring the polynomial ring $\mathbb{Z}/(2)[\lambda, w_2, w_3, \ldots, w_{2n+1}]$ where for $2 \leq i \leq 2n+1$, the elements $w_i \in H^i(BSO(2n+1); \mathbb{Z}/(2))$ are the Stiefel-Whitney classes.

In this paper, we consider the even case $GO(2n)$. The main result is an explicit determination, in terms of generators and relations, of the singular cohomology ring $H^*(BGO(2n); \mathbb{Z}/(2))$. This is the Theorem 3.9 below.

An outline of the argument is as follows.

To each action of the group $\mathbb{C}^*$ on any space X, we functorially associate a certain derivation $s : H^*(X) \to H^*(X)$ on the cohomology ring of X, which is graded of degree -1, with square zero (see Lemma 2.2 below). In terms of the action $\mu : \mathbb{C}^* \times X \to X$ and the projection $p : \mathbb{C}^* \times X \to X$, it is given by the formula

$$\mu^* - p^* = \eta \otimes s$$

where η is the positive generator of $H^1(\mathbb{C}^*)$.

When X, together with its given $\mathbb{C}^*$-action, is the total space of a principal $\mathbb{C}^*$-bundle $\pi : X \to Y$ over some base Y, recall that we have a long exact Gysin sequence

$$\cdots \xrightarrow{\lambda} H^i(Y) \xrightarrow{\pi^*} H^i(X) \xrightarrow{d} H^{i-1}(Y) \xrightarrow{\lambda} H^{i+1}(Y) \xrightarrow{\pi^*} \cdots$$

We show (Lemma 2.3) that in this case, the derivation s on $H^*(X)$ equals the composite

$$s = \pi^* \circ d$$

As $O(2n) \subset GO(2n)$ is normal with quotient $\mathbb{C}^*$, $BO(2n)$ can be regarded as the total space of a principal $\mathbb{C}^*$-bundle $BO(2n) \to BGO(2n)$. The resulting action of $\mathbb{C}^*$ on $BO(2n)$ gives a derivation s on the ring $H^*(BO(2n)) = \mathbb{Z}/(2)[w_1, \ldots, w_{2n}]$ with $\mu^* - p^* = \eta \otimes s$. The last equality enables us to write the following expression for s (see Lemma 3.4 below)

$$s = \sum_{i=1}^{n} w_{2i-1} \frac{\partial}{\partial w_{2i}}$$

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