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THE EQUIVALENCE OF HILBERT AND MUMFORD STABILITY FOR VECTOR BUNDLES*

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Abstract. We prove the equivalence of the notions of Hilbert (semi)stability and Mumford (semi)stability for vector bundles on smooth curves for arbitrary rank.

Introduction. Let d, g, and r be fixed positive integers, W a complex vector space of dimension $p := d + r(1-g)$, and $\mathfrak{G} = G(W,r)$ the Grassmannian of r-dimensional quotients of W. On $\mathfrak{G}$, there is the universal quotient

$$W \otimes \mathcal{O}_{\mathfrak{G}} \longrightarrow E_{\mathfrak{G}}$$

which induces a surjection $\bigwedge^r W \otimes \mathcal{O}_{\mathfrak{G}} \longrightarrow \bigwedge^r E_{\mathfrak{G}}$, defining the Pluecker embedding $\mathfrak{G} \hookrightarrow \mathbb{P}(\bigwedge^r W)$. Now, let $C \hookrightarrow \mathfrak{G}$ be a smooth curve of genus g such that $E_C := E_{\mathfrak{G}|C}$ has degree d. Then, C gets embedded into $\mathbb{P}(\bigwedge^r W)$ as a curve with Hilbert polynomial $P(m) = \chi((\bigwedge^r E_C)^{\otimes m}) = dm + (1-g)$. From the restriction $W \otimes \mathcal{O}_C \longrightarrow E_C$ of the universal quotient to C we derive homomorphisms $\bigwedge^r W \otimes \mathcal{O}_C \longrightarrow \bigwedge^r E_C$, and for all $m \geq 1$

$$\psi_C^m \colon S^m \bigwedge^r W \longrightarrow H^0\big((\bigwedge^r E_C)^{\otimes m}\big).$$

If ψ_C^m is surjective (as will be the case for large m) and $h^0((\bigwedge^r E_C)^{\otimes m}) = P(m)$, this yields

$$\varphi_C^m := \bigwedge^{P(m)} \psi_C^m \colon \bigwedge^{P(m)} (S^m \bigwedge^r W) \longrightarrow \mathbb{C}.$$

We call C or, abusively, E_C *m-Hilbert (semi/poly)stable*, if ψ_C^m is surjective, $h^0((\bigwedge^r E_C)^{\otimes m}) = P(m)$, and the point $[\varphi_C^m]$ in $\mathbb{P}(\bigwedge^{P(m)} (S^m \bigwedge^r W))$ is (semi/poly)stable w.r.t. the natural action of $\mathrm{SL}(W)$ on that space, and *Hilbert (semi/poly)stable*, if it is m-Hilbert (semi/poly)stable for all m sufficiently large. This is now a new stability concept for the vector bundle E_C entering in competition to classical Mumford stability. It goes back to Gieseker and Morrison ([4], [5]). Its main motivation is to obtain an alternative compactification, called *Hilbert stable compactification* by Teixidor [11], of the universal moduli space of semistable vector bundles of rank r and degree d over $\mathfrak{M}_g$, the moduli space of smooth curves of genus g, by letting C vary and degenerate in $\mathfrak{G}$. In contrast to the *slope stable compactification* of Pandharipande [9] which involves torsion free sheaves on singular curves, this compactification would take place entirely in the realm of vector bundles. Its potential usefulness is illustrated by the paper [5] where Hilbert stable vector bundles on a nodal curve are used to prove a conjecture of Newstead and Ramanan on the moduli space of stable rank two bundles over a smooth curve. In order to make such a theory

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