ASIAN J. MATH.
Vol. 5, No. 1, pp. 43–78, March 2001

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CALABI-YAU THREEFOLDS OF QUOTIENT TYPE*

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Dedicated to Professor V. A. Iskovskih on the occasion of his sixtieth birthday

Abstract. By a Calabi-Yau threefold we mean a minimal complex projective threefold X such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. This paper consists of four parts. In Section 1 we formulate an equivariant version of Torelli Theorem of K3 surfaces with finite group action and deduce some more geometrical consequences. In Section 2, we classify Calabi-Yau threefolds with infinite fundamental group by means of their minimal splitting coverings introduced by Beauville, and deduce as its Corollary that the nef cone is a rational simplicial cone and any rational nef divisor is semi-ample provided that $c_2(X) \equiv 0$ on $\mathrm{Pic}(X)_{\mathbb{R}}$. We also derive a sufficient condition for $\pi_1(X)$ to be finite in terms of the Picard number in an optimal form. In Section 3, we give a fairly concrete structure Theorem concerning c_2-contractions of Calabi-Yau threefolds as a generalisation and also a correction of our earlier works for simply connected ones. In Section 4, applying the results in these three sections together with Kawamata's finiteness result of the relatively minimal models of a Calabi-Yau fiber space, we show the finiteness of the isomorphism classes of c_2-contractions of each Calabi-Yau threefold. As a special case, we find the finiteness of abelian pencil structures on each X up to $\mathrm{Aut}(X)$.

0. Introduction. In the light of the minimal model theory, we define *a Calabi-Yau threefold* to be a $\mathbb{Q}$-factorial terminal projective threefold X defined over $\mathbb{C}$ such that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$, and regard the second Chern class $c_2(X)$ as a linear form on $\mathrm{Pic}(X)_{\mathbb{R}}$ through the intersection pairing, where $c_2(X)$ for a singular X is defined as $c_2(X) := \nu_*(c_2(\tilde{X}))$ via a resolution $\nu : \tilde{X} \to X$ and is known to be well-defined (see for example [Og1, Lemma (1.4)]).

However, as is pointed out by several authors, this preferable definition of Calabi-Yau threefold has an inevitable defect: Those Calabi-Yau threefolds, such as Igusa's example ([Ig, Page 678], [Ue, Example 16.16]), that are given as an étale quotient of an abelian threefold are then included in our category. We call them *of Type A*. Indeed, their pathological nature sometimes prevents us from studying Calabi-Yau threefolds uniformly. For instance,

(1) there are no rational curves on Calabi-Yau threefolds of Type A, while it is expected, and has been already checked in some extent, that most of Calabi-Yau threefolds contain rational curves (see [Wi1], [HW] and [EJS]);
(2) $c_2(X) = 0$ for such X but $c_2(X) \neq 0$ for others.

Here, for the last statement, we recall the following result due to S. Kobayashi in the smooth case and Shepherd-Barron and Wilson in the general case:

THEOREM ([KB, CHAP. IV, COROLLARY (4.15)], [SBW, COROLLARY]). *Let X be a Calabi-Yau threefold. Then, X is of Type A if and only if $c_2(X) = 0$.* □

One of the main purposes of this paper is to compensate for this defect by revealing explicit geometric structures of Calabi-Yau threefolds of Type A. It will turn out that they are remarkably few so that one can in principle handle them separately in case. Our result is:

* Received January 13, 2000; accepted for publication February 9, 2000.
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