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ON HESSIAN RIEMANNIAN STRUCTURES*

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Abstract. In Proposition 4.1 a characterization is given of Hessian Riemannian structures in terms of a natural connection in the general linear group $\mathrm{GL}(n, \mathbf{R})^+$, which is viewed as a principal $\mathrm{SO}(n)$-bundle over the space of positive definite symmetric $n \times n$-matrices. For $n = 2$, Proposition 5.3 contains an interpretation of the curvature of a Hessian Riemannian structure at a given point, in terms of an umbilic point of a related surface in $\mathbf{R}^3$.

0. Introduction. In convex programming, one makes use of a so-called self--concordant barrier function f on an open convex subset Q of $\mathbf{R}^n$, cf. the book [7] of Nesterov and Nemirovskii, and one is interested in the behaviour of the geodesics of the Riemannian structure defined by the Hessian of f. I got acquainted with the subject when I was asked to give an introduction to Riemannian geometry at the conference HPOPT'99 at the Erasmus University Rotterdam, in June 1999.

The formula for the curvature tensor of a Hessian Riemannian structure, cf. (1.7) below, involves only second and third order derivatives of the function f, and no fourth order ones as one would a priori expect. In my attempt to understand this, I arrived at the characterization of Hessian Riemannian structures in Proposition 4.1. In the case $n = 2$ there is also an interpretation of the curvature of a Hessian Riemannian structure in terms of umbilic points of surfaces, see Section 5.

The study of Hessian Riemannian structures on convex domains goes back at least to Koszul [6] and Vinberg [11], who were inspired by the theory of bounded domains in $\mathbf{C}^n$ with its Bergmann metric. Closely related to our subject is Shima's theory of Hessian manifolds, cf. [10]. Ruuska [8] characterized Hessian Riemannian structures as those which admit an abelian Lie algebra of gradient vector fields, where the local action is simply transitive. Hitchin [4] characterized Hessian Riemannian structures in term of a Lagrangean submanifolds of the cotangent bundle. I am grateful to Nigel Hitchin and Lieven Vanhecke for getting me started with the literature on Hessian Riemannian structures.

As a general reference on differential geometry one may use [5].

1. Hessian Riemannian Structures. Suppose that f is a smooth strongly convex function, defined on an open subset Q of $\mathbf{R}^n$. The strong convexity of f means that for every $x \in Q$ the Hessian $\partial_i \partial_j f(x)$ is positive definite, which implies that $g_{ij}(x) := \partial_i \partial_j f(x)$ defines a Riemannian structure. Here, and in the sequel, we use the abbreviation $\partial_i \phi(x)$ for the partial derivative $\partial \phi(x)/\partial x^i$ of any function ϕ with respect to the i-th coordinate.

For a general Riemannian structure $g_{ij}(x)$ the inverse matrix is denoted by $g^{kl}(x)$. The Christoffel symbols then are defined by

$$\Gamma_{ij}^k(x) := \sum_{l=1}^n g^{kl}(x)\, \Gamma_{lij}(x), \tag{1.1}$$

in which

$$\Gamma_{lij}(x) := \frac{1}{2}\left[\partial_i g_{jl}(x) - \partial_l g_{ij}(x) + \partial_j g_{li}(x)\right]. \tag{1.2}$$

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