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THE CLASSIFICATION OF HOMOGENEOUS 2-SPHERES IN $\mathbb{C}P^n$ *

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Dedicated to Professor Shiing-shen Chern at the occasion of his 90th birthday

Abstract. Let M be a connected, orientable surface. An immersion $x : M \to \mathbb{C}P^n$ is called homogeneous, if for any two points $p, q \in M$ there exists a holomorphic isometry T of $\mathbb{C}P^n$ and a diffeomorphism $\sigma : M \to M$ such that $\sigma(p) = q$ and $x \circ \sigma = T \circ x$. All such T form a subgroup G of the holomorphic isometry group $U(n+1)/S^1$ and $x(M)$ is an orbit surface of G. Such surfaces in $\mathbb{C}P^n$ have constant curvature, constant Kaehler angle, and in general are non-minimal. In this paper we show that any homogeneous surface in $\mathbb{C}P^n$ generates a sequence of homogeneous surfaces in $\mathbb{C}P^n$. In the case of a homogeneous sphere $x : S^2 \to \mathbb{C}P^n$ the sequence has to stop in the both directions, and any two different homogeneous spheres in the sequence are complex orthogonal. We give a construction of homogeneous 2-spheres in $\mathbb{C}P^n$ by using Veronese sequences in $\mathbb{C}P^m$ with $m \leq n$ as foundation stones, and prove that any homogeneous 2-sphere in $\mathbb{C}P^n$ can be obtained (up to a holomorphic isometry) by this construction.

1. Introduction. A surface in $\mathbb{C}P^n$ is said to be homogeneous, if it is a 2-dimensional orbit of a subgroup in the holomorphic isometry group $U(n+1)/S^1$ of $\mathbb{C}P^n$. Standard examples of homogeneous 2-spheres are the so-called Veronese sequence in $\mathbb{C}P^n$, which are also minimal in $\mathbb{C}P^n$ (see, for examples, [B-J-R-W], [B-O] and [B-W]).

As is well-known, any surface in $\mathbb{C}P^n$ sits in a sequence of surfaces in $\mathbb{C}P^n$, which can be constructed by using derivatives with respect to its complex coordinate z and $\bar{z}$. Surfaces in the sequence may contain singularities. If $x : M \to \mathbb{C}P^n$ is minimal (harmonic), then all surfaces in the sequence are minimal (harmonic), and the sequence is called the harmonic sequence of x. In this paper we show that any homogeneous surface in $\mathbb{C}P^n$ generates a sequence of homogeneous surfaces. In case of homogeneous 2-sphere we show that any two different homogeneous spheres in its homogeneous sequence are complex orthogonal, thus its homogeneous sequence has to stop in both directions.

Using the Veronese sequence in $\mathbb{C}P^k, k = 0, 1, \cdots$, we can give a complete construction of homogeneous 2-spheres in $\mathbb{C}P^n$. For each integer $n \geq 0$ we denote by

$$\{[\phi_n^0], [\phi_n^1], \cdots, [\phi_n^n]\} \tag{1.1}$$

the Veronese sequence in $\mathbb{C}P^n$ (see §4). We make the convention that $[\phi_0^0] = [1]$ and the sequence $[\phi_0^1], [\phi_1^1] : \mathbb{C}P^1 \to \mathbb{C}P^1$ are defined by $[\phi_0^1] = id$ and $[\phi_1^1]([z, w]) = [-\bar{w}, \bar{z}]$. Let $\{n_1, n_2, \cdots, n_r\}$ be nonnegative integers. We take from each Veronese sequence in $\mathbb{C}P^{n_\alpha}$ a surface $[\phi_{j_\alpha}^{n_\alpha}]$ such that

$$n_1 - 2j_1 = n_2 - 2j_2 = \cdots = n_r - 2j_r\,. \tag{1.2}$$

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