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WEAK $SPIN(9)$-STRUCTURES ON 16-DIMENSIONAL RIEMANNIAN MANIFOLDS*

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Abstract. The aim of the present paper is the investigation of $Spin(9)$-structures on 16-dimensional manifolds from the point of view of topology as well as holonomy theory. First we construct several examples. Then we study the necessary topological conditions resulting from the existence of a $Spin(9)$-reduction of the frame bundle of a 16-dimensional compact manifold (Stiefel-Whitney and Pontrjagin classes). We compute the homotopy groups $\pi_i(X^{84})$ of the space $X^{84} = SO(16)/Spin(9)$ for $i \leq 14$. Next we introduce different geometric types of $Spin(9)$-structures and derive the corresponding differential equation for the unique self-dual 8-form Ω^8 assigned to any type of $Spin(9)$-structure. Finally we construct the twistor space of a 16-dimensional manifold with $Spin(9)$-structure and study the integrability conditions for its universal almost complex structure as well as the structure of the holomorphic normal bundle.

1. Introduction. The aim of this paper is to present a weak holonomy concept associated to the Lie group $Spin(9)$. The spin representation of the group $Spin(9)$ is real and 16-dimensional. According to Berger's holonomy theorem, $Spin(9)$ can occur as the holonomy group of a 16-dimensional Riemannian manifold. However, D. Alekseevski (see [2]) and R. Brown/ A. Gray (see [6]) proved that any complete 16-dimensional Riemannian manifold whose holonomy group is contained in $Spin(9)$ is necessarily flat or isometric to the Cayley plane $F_4/Spin(9)$ or its non-compact dual $F_4^*/Spin(9)$.

In 1971 A. Gray introduced the concept of weak holonomy. He proved that if a manifold has one of the groups

$$G = SO(n), \quad SU(n), \quad Sp(n)\cdot Sp(1), \quad Sp(n)\cdot SO(2), \quad Sp(n), \quad Spin(7)$$

as weak holonomy group, then its holonomy is in fact already contained in G. Consequently, only 3 groups may admit a weak holonomy concept that is more general than the traditional holonomy approach (see [14]):

$$G = U(n), \quad G = G_2 \text{ in dimension } 7, \quad G = Spin(9) \text{ in dimension } 16.$$

The first two cases yield a rich geometric structure both as weak and as classical holonomy groups and have been studied intensively. Manifolds with weak holonomy group $U(n)$ are called nearly Kähler (see [13]). A. Gray has investigated them since 1976 (see [15]) and pointed out that they have special properties in dimension 6. This effect is closely related to the fact that, on a 6-dimensional manifold, the existence of a nearly Kähler structure is equivalent to the existence of a real Killing spinor (see [17]). In 1981 S. Marchiafava (see [21]) characterized 7-dimensional manifolds with weak holonomy group G_2, and M. Fernandez / A. Gray (see [9]) studied the different geometric types of G_2-structures systematically. In particular, nearly parallel G_2-structures correspond again to real Killing spinors (see [12]). Only the case of weak holonomy $Spin(9)$ on 16-dimensional manifolds has been neglected until now.

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