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THE STRUCTURE EQUATIONS OF A COMPLEX FINSLER MANIFOLD*

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Abstract. For a strongly pseudo-convex complex Finsler manifold M, a bundle $U_F(M)$ of adapted unitary frames is canonically defined. A non-linear Hermitian connection on $U_F(M)$, invariant under local biholomorphic isometries, is given and it proved to be unique. By means of such connection, an absolute parallelism on $U_F(M)$ is determined and a new set of structure functions which generate all the isometric invariants of a Finsler metric is obtained.

A pseudo-convex complex Finsler manifolds M, which admits a totally geodesic complex curve with a given constant holomorphic sectional curvature through any point and any direction, is called *E-manifold*. Main examples of E-manifolds are the smoothly bounded, strictly convex domains in $\mathbb{C}^n$, endowed with the Kobayashi metric. A complete characterization of E-manifolds, using the previously defined structure functions, is given and a smaller set of generating functions for the isometric invariants of E-manifolds is determined.

1. Introduction. The main purpose of this paper is to give a complete set of invariants, which characterize a strongly pseudoconvex complex Finsler metric up to local biholomorphic isometries. Several properties of these invariants are immediately related with the intrinsic geometry of the Kobayashi metric of the smoothly bounded, strongly convex domains in $\mathbb{C}^n$.

Let M be a complex manifold and J its complex structure. The well-known *infinitesimal Kobayashi pseudo-distance* k_M *on* $\tilde{T}M = TM \setminus \{\text{zero section}\}$ can be defined as follows ([Ko]): for any $x \in M$ and any $0 \neq v \in T_xM$, let $\mathcal{A}_v$ the set of all $r \in \mathbb{R}^+$ such that there exists a holomorphic map $f : \Delta_r \to M$ from $\Delta_r = \{|z| < r\} \subset \mathbb{C}$ into M with $f(0) = x$ and $f_*(\frac{\partial}{\partial\zeta}) \in \mathbb{C}v$. Then

$$k_M(v) = \inf_{r \in \mathcal{A}(v)} \frac{1}{r}.$$

We consider the following class of complex manifolds.

DEFINITION 1.1. A complex manifold (M, J) is a *Lempert manifold* if

(1) the infinitesimal Kobayashi pseudo-distance k_M is a strongly pseudoconvex Finsler metric, that is:

a) it is a smooth function on $\tilde{T}M$ with values in $\mathbb{R}^+$;

b) $k_M(\lambda v) = |\lambda| k_M(v)$ for any $\lambda \in \mathbb{C}^*$ and $v \in \tilde{T}M$;

c) at any point $x \in M$ the hypersurface $S_x = \{v \in T_xM \ : k_M(v) = 1\}$ is strongly pseudoconvex in T_xM;

(2) for any non-vanishing complex vector $w \in T^{\mathbb{C}}_xM \subset T^{\mathbb{C}}M$, there exists a complex curve $\gamma_w : U \subset \mathbb{C} \to M$, such that $\gamma_w(0) = x$, $\gamma'_w(0) = w$ and $\gamma(U)$ is a totally geodesic submanifold of M;

(3) the metric, which is induced by k_M on the totally geodesic complex curve $\gamma_w(U)$, is Kähler and with constant holomorphic curvature equal to -4;

(4) the (finite) Kobayashi distance d_M, determined by k_M, is complete and the exponential map $\exp : T_xM \to M$ is a diffeomorphism for any $x \in M$.

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