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CONVEX POLYHEDRA IN LORENTZIAN SPACE-FORMS*

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Abstract. Aleksandrov [Ale51] characterized the metrics induced on convex polyhedra in E^3, H^3 and S^3. We give analogs for compact and complete polyhedra in Lorentzian space-forms.

There are three types of convex polyhedra in the de Sitter space S^3_1. One, which includes generalized hyperbolic polyhedra, was treated in [Sch98a]. For the second, we characterize the induced metrics, and show that each is obtained on a unique polyhedron satisfying a natural condition at infinity. For the last type — compact polyhedra bounding compact domains — we describe the induced metrics, and give an existence and uniqueness result for a smaller class of metrics.

The results on complete polyhedra are consequences of the study of the metrics induced on convex polyhedra in a natural extension of H^3 by S^3_1. We also characterize the metrics induced on compact, convex polyhedra in the Minkowski space E^3_1.

Those description are partly similar to those obtained in the Riemannian cases, but they also involve new elements of a metric and combinatorial nature.

Résumé. Aleksandrov [Ale51] a caractérisé les métriques induites sur les polyèdres convexes dans E^3, H^3 et S^3. On donne des résultats similaires pour les polyèdres compacts ou complets dans les formes d'espace lorentziennes.

On distingue trois types de polyèdres convexes dans l'espace de Sitter S^3_1. L'un d'eux, incluant les polyèdres hyperboliques généralisés, est étudié dans [Sch98a]. Pour le second, on caractérise les métriques induites, et on montre que chacune est obtenue sur un unique polyèdre satisfaisant une conditions naturelle à l'infini. Pour le troisième type — les polyèdres compacts bordant des domaines compacts — on donne une description des métriques induites, et un résultat d'existence et d'unicité pour une classe plus restreinte de métriques.

Les résultats concernant les polyèdres complets sont des conséquences de l'étude des métriques induites sur les polyèdres convexes dans une extension naturelle de H^3 par S^3_1. On caractérise les aussi les métriques induites sur les polyèdres convexes dans l'espace de Minkowski E^3_1.

Ces descriptions sont partiellement analogues à celles obtenues dans le cas riemannien, mais elles font aussi intervenir de nouveaux éléments de nature métrique et combinatoire.

1. Introduction and main results. The metrics induced on convex polyhedra of 3-dimensional Riemannian space-forms are completely described by the following well-known theorem:

THEOREM 1.1 (Aleksandrov [Ale51]). *Choose $K_0 \in \{-1, 0, 1\}$. A Riemannian metric g on S^2 with conical singularities is induced by a convex polyhedral embedding in the simply connected space M_{K_0} with constant curvature K_0 if and only if g has constant curvature K_0 except at a finite number of singular points $x_1, \cdots, x_N$, where the singular curvature is positive. The embedding is then unique modulo global isometries.*

This paper intends to give similar results for compact polyhedra in two Lorentzian space-forms, the de Sitter and the Minkowski spaces. We will also describe the induced metrics on complete polyhedra in the de Sitter space, along the lines of [Sch98a] for complete hyperbolic polyhedra.

Note that a basic reason to study the metrics induced on convex polyhedra in the de Sitter space is that there is a classical duality, in particular between convex polyhedra in H^3 and in S^3_1, which exchanges the edge lengths and the dihedral angles (this is recalled in section 2). The dihedral angles of a convex hyperbolic polyhedron are therefore related to the induced metric on its dual, which is a convex polyhedron

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