ASIAN J. MATH.
Vol. 5, No. 2, pp. 365–378, June 2001
© 2001 International Press
004

WILLMORE HYPERSURFACES IN A SPHERE*

HAIZHONG LI†

Abstract. Let $x : M \to S^{n+1}$ be an n-dimensional hypersurface in S^{n+1}, $x : M \to S^{n+1}$ is called a Willmore hypersurface if it is an extremal hypersurface to the following Willmore functional:

$$\int_M (S - nH^2)^{\frac{n}{2}} dv,$$

where $S = \sum_{i,j}(h_{ij})^2$ is the square of the length of the second fundamental form, H is the mean curvature of M. In this paper, through study of the Euler-Lagrange equation of the Willmore functional, we obtain an integral inequality of Simons' type for Willmore hypersurfaces in S^{n+1} and give a characterization of *Willmore tori* by use of our integral formula. We also classify all isoparametric Willmore hypersurfaces in S^{n+1}.

1. Introduction. Let M be an n-dimensional compact hypersurface of the $(n+1)$-dimensional unit sphere S^{n+1}. If h_{ij} denotes the components of the second fundamental form of M, S denotes the square of the length of the second fundamental form and H denotes the mean curvature, then we have

$$S = \sum_{i,j}(h_{ij})^2, \quad \mathbf{H} = \frac{1}{n}\sum_k h_{kk} e_{n+1}, \quad H = |\mathbf{H}|,$$

where e_{n+1} is an unit normal vector field of M in S^{n+1}.

We define the following non-negative function on M

$$\rho^2 = S - nH^2, \tag{1.1}$$

which vanishes exactly at the umbilic points of M.

Willmore functional is the following non-negative functional (see [4], [20] or [22])

$$\int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv.$$

It was shown in [4] and [20] that this functional is an invariant under Moebius (or conformal) transformations of S^{n+1}. We use the term Willmore hypersurfaces to call its critical points, because when $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces.

In this paper, we first prove the following theorem

THEOREM 1. *Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. Then M is a Willmore hypersurface if and only if*

$$\begin{aligned} &-\rho^{n-2}(2HS - nH^3 - \sum_{i,j,k} h_{ij}h_{jk}h_{ki}) \\ &+(n-1)\Delta(\rho^{n-2}H) - \sum_{i,j}(\rho^{n-2})_{,ij}(nH\delta_{ij} - h_{ij}) = 0, \end{aligned} \tag{1.2}$$

*Received May 3, 2000; accepted for publication May 13, 2000.

†Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China (hli@math.tsinghua.edu.cn) and Department of Mathematics, Harvard University, Cambridge MA, 02138, USA (hli@math.harvard.edu). The author is partially supported by the project No.19701017 of NSFC and US-China Cooperative Research NSF Grant INT-9906856.