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BIALGEBRA AND GEOMETRY OF PLANE QUARTICS*

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Abstract. Weak semisimplicity of the bialgebra of ternary quartic forms is proved.

In fact there being no direct method of proceeding from the invariant to the geometric meaning ...

If it be not obtained we should console ourselves with the reflexion that the uninterpreted forms are of little geometrical interest in the present state of knowledge;

Besides if we regard the algebra as being merely helpful to geometry in the analytical formulations of results, it does not follow that everything in the algebra need be taken seriously from the geometrical point of view.

(From No. 229 of "The algebra of Invariants" by J. H. Grace and A. Young, Cambridge, 1903)

1. First definitions and introduction.

(1.0) Conventions. In the article, the ground field K is always of characteristic zero and algebraically closed.

Let W be a finite-dimensional K-vector space, W^* be the dual space of W. Sometimes, elements of W (denoted as $a, b, c...$) will be called linear forms, elements of W^* (denoted as $p, q, r...$) will be called vectors. Therefore, elements of $\mathbb{P}(W)$ are hyperplanes (or lines, if W is three-dimensional), elements of $\mathbb{P}(W^*)$ are points. Also, we will use geometric term *pencil* for two-dimensional subspaces of W (or W^*). If $\{a, b\}$ is a basis of a pencil P, then we will denote the pencil P by $(\lambda a + \mu b)$ or by $(Ka + Kb)$. Let

$$W \times W^* \to K,\ (a, p) \mapsto < a, p > \in K,$$

$$W^* \times W \to K,\ (p, a)\ \mapsto < p, a > \in K$$

be two natural pairings denoted with the same symbol $<,>$.

We will say that two bases $\{e_1, ..., e_n\} \subset W,\ \ \{f_1, ..., f_n\} \subset W^*$ are projectively dual , if $< e_i, f_j >= 0$ for $i \neq j,\ < e_i, f_i > \neq 0$. Here, projective point of view means that we do not try to normalize elements of the bases and consider them up to proportionality.

(1.1) Definition. *Bialgebra.* Bimultiplication law on W (defining a bialgebra structure on W) consists of two bilinear maps (both of which will be denoted by square brackets)

$$W \times W \to W^*, \qquad (a, b) \mapsto [a, b] \in W^*,$$

$$W^* \times W^* \to W, \qquad (p, q) \mapsto [p, q] \in W.$$

The number $\dim(W)$ is dimension (or rank) of the bialgebra $(W, [\ ,\ ])$.

The number $(\dim(W) - 1)$ is projective dimension of the bialgebra.

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