ASIAN J. MATH.
Vol. 5, No. 3, pp. 433–440, September 2001

REGULARITY OF BUTTERWORTH REFINABLE FUNCTIONS*

AIHUA FAN† AND QIYU SUN‡

Abstract. Let Ψ_N be the refinable function with Butterworth filter $\cos^{2N}\frac{\xi}{2}\big(\cos^{2N}\frac{\xi}{2}+\sin^{2N}\frac{\xi}{2}\big)^{-1}$ and let $s_p(\Psi_N)$ be the Fourier exponent of Ψ_N of order p $(0<p\le\infty)$. It is proved that

$$0\le s_\infty(\Psi_N)-N\frac{\ln 3}{\ln 2}\le\frac{\ln(1+3^{-N})}{\ln 2}\qquad (N\ge 1)$$

and for $0<p<\infty$

$$-\frac{\ln(1+r_0^{Np})}{p\ln 2}\le s_p(\Psi_N)-N\frac{\ln 3}{\ln 2}\le\frac{\ln(1+3^{-N})}{\ln 2}\qquad (N\ge 1)$$

where $r_0\in(0,1)$ is independent of p and N.

1. Introduction and Result. In this paper we study the solutions of some *refinement equations* of the form

$$\phi(x)=\sum_{j\in\mathbb{Z}}c_j\phi(2x-j)\qquad (x\in\mathbb{R}) \tag{1.1}$$

where the coefficients c_j are supposed to satisfy the arithmetic condition $\sum_{j\in\mathbb{Z}}c_j=2$ and the exponential decay condition $|c_j|\le Ce^{-\beta|j|}$ ($C,\beta>0$ constants). Solutions of a refinement equation are called *refinable functions.* The 2π-periodic function

$$m(\xi)=\frac{1}{2}\sum_{j\in\mathbb{Z}}c_je^{-ij\xi}$$

is called the *filter* of the refinement equation (1.1). A continuous function ϕ is called a *cardinal interpolant* if $\phi(0)=1$ and $\phi(k)=0$ for all nonzero integer k. It is known that there is an important class of refinable functions which are cardinal interpolants and whose filters satisfy

$$m(\xi)+m(\xi+\pi)=1. \tag{1.2}$$

Such a filter $m(\xi)$ can be put into the factorized form

$$m(\xi)=\left(\frac{1+e^{-i\xi}}{2}\right)^N R(\xi) \tag{1.3}$$

where N is a strictly positive integer and $R(\xi)$ is a 2π-periodic function whose Fourier coefficients decay exponentially. The minimal degree solution of (1.2) having the factorized form (1.3) is given by

$$m_N(\xi)=\cos^{2N}\frac{\xi}{2}\sum_{s=0}^{N-1}\binom{N-1+s}{s}\sin^{2s}\frac{\xi}{2}.$$

*Received April 8, 1999; accepted for publication June 19, 2000.
†Département de Mathématiques et Informatique, Université de Picardie Jules Verne, 33 Rue Saint Leu, 80039 Amiens Cedex 1, France.
‡Department of Mathematics, National University of Singapore, 10 Kent Ridge Road, Singapore 119260 (matsunqy@leonis.nus.edu.sg).