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A GENERALIZED INDEX THEOREM FOR MORSE-STURM SYSTEMS AND APPLICATIONS TO SEMI-RIEMANNIAN GEOMETRY*

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Abstract. We prove an extension of the Index Theorem for Morse–Sturm systems of the form $-V'' + RV = 0$, where R is symmetric with respect to a (non positive) symmetric bilinear form, and thus the corresponding differential operator is not self-adjoint. The result is then applied to the case of a Jacobi equation along a geodesic in a Lorentzian manifold, obtaining an extension of the Morse Index Theorem for Lorentzian geodesics with variable initial endpoints. Given a Lorentzian manifold $(\mathcal{M}, g)$, we consider a geodesic γ in $\mathcal{M}$ starting orthogonally to a smooth submanifold $\mathcal{P}$ of $\mathcal{M}$. Under suitable hypotheses, satisfied, for instance, if $(\mathcal{M}, g)$ is stationary, the theorem gives an equality between the index of the second variation of the action functional f at γ and the sum of the *Maslov index* of γ with the index of the metric g on $\mathcal{P}$. Under generic circumstances, the Maslov index of γ is given by an algebraic count of the $\mathcal{P}$-focal points along γ. Using the Maslov index, we obtain the global Morse relations for geodesics between two fixed points in a stationary Lorentzian manifold.

1. Introduction. The goal of this paper is to prove an index theorem for Morse–Sturm systems of differential equations with coefficients that are symmetric with respect to an indefinite inner product of $\mathbb{R}^n$. The main motivation for this kind of investigation comes from semi-Riemannian geometry, where Morse–Sturm systems appear in the form of *Jacobi* equations for vector fields along geodesics.

Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold, $\mathcal{P}$ a smooth submanifold of $\mathcal{M}$ and $\gamma : [0, 1] \mapsto \mathcal{M}$ be a geodesic in $\mathcal{M}$, with $\gamma(0) \in \mathcal{P}$ and $\dot\gamma(0) \in T_{\gamma(0)}\mathcal{P}^\perp$; set $q = \gamma(1)$. The curve γ is then a stationary point of the *action* functional

$$f(z) = \frac{1}{2}\int_0^1 g(\dot z, \dot z)\,\mathrm{d}t,$$

defined in the space $\Omega_{\mathcal{P},q}$ of curves joining $\mathcal{P}$ and the point q in $\mathcal{M}$. The *index form* $I_{\{\gamma,\mathcal{P}\}}$ is the symmetric bilinear form given by the second variation of f, defined on the tangent space $T_\gamma\Omega_{\mathcal{P},q}$, which consists of vector fields V along γ with $V(0) \in T_{\gamma(0)}\mathcal{P}$ and $V(1) = 0$. We recall the definition of $I_{\{\gamma,\mathcal{P}\}}$:

$$I_{\{\gamma,\mathcal{P}\}}(V, W) = \int_0^1 \Big[g(\nabla_{\dot\gamma}V, \nabla_{\dot\gamma}W) + g(R(\dot\gamma, V)\,\dot\gamma, W)\Big]\,\mathrm{d}t - S_{\dot\gamma(0)}(V(0), W(0)), \tag{1.1}$$

where ∇ is the covariant derivative of the Levi–Civita connection of g, R is the curvature tensor of ∇ and $S_{\dot\gamma(0)}$ is the second fundamental form of $\mathcal{P}$ in the direction of $\dot\gamma(0)$.

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